Linear Algebra – Day 30

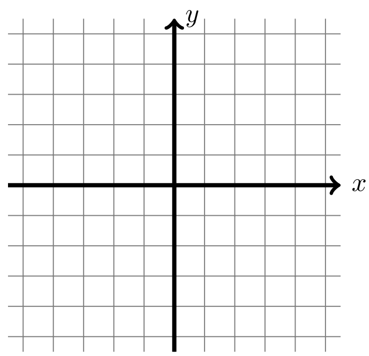
MATH 220

For Problems 1 and 2, let $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

1. Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$. How do your results compare?

2. (a) Plot vector $\mathbf{u}$ on a coordinate plane and find its length. Do the same for $\mathbf{v}$.

☞ Notation: The length of a vector $\mathbf{u}$ is denoted $||\mathbf{u}||$.



(b) Compute the dot product $\mathbf{u} \cdot \mathbf{u}$.

(c) How are your answers in parts (a) and (b) related?

(d) Now draw a random vector $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$. Can you express the length of $\mathbf{x}$ using the dot product?

3. Find the vector $\mathbf{w}$ with the following properties:

- the length of $\mathbf{w}$ is 1, and
- $\mathbf{w}$ is in the same direction as $\mathbf{v}$.

☞ HINT: If a vector had a length of 3 and you wanted to make it have length 1 but not change the direction, what would you do to it?

4. **Leo:** I just love these two matrices: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Cleo: Let's multiply A and B together!

Leo: You're way too enthusiastic about this. But OK. Here it is:

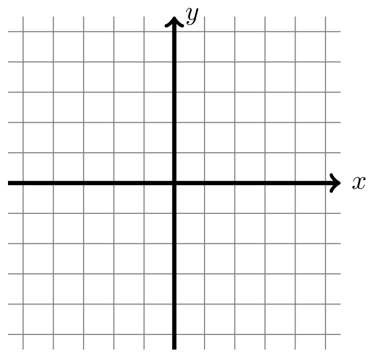
$$AB = \begin{bmatrix} a+2d+3g & b+2e+3h & c+2f+3i \\ 4a+5d+6g & 4b+5e+6h & 4c+5f+6i \\ 7a+8d+9g & 7b+8e+9h & 7c+8f+9i \end{bmatrix}$$

Cleo: OH WOW...there are dot products *everywhere*!

Group chat: Discuss Cleo's observation. In general, how can dot products be used to find the entry in the ith row and jth column of a matrix product AB?

5. Now let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

(a) Plot vectors $\mathbf{u}$ and $\mathbf{v}$ and verify geometrically that they are perpendicular to each other.



(b) Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.

(c) Plot a few more vectors that are perpendicular to $\mathbf{u}$. Geometrically, what does the collection of *all* vectors that are perpendicular to $\mathbf{u}$ look like?

(d) Suppose $\mathbf{w}$ is any vector that is perpendicular to $\mathbf{u}$. What can you say about the dot product $\mathbf{u} \cdot \mathbf{w}$? Explain your reasoning.

6. Let H be the collection of all vectors $\mathbf{x}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \mathbf{x} = 0$.

☞ That is, H is all the vectors that are orthogonal (perpendicular) to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Verify that the vector $\begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$ is in H.

(b) *True or False*: $\begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$ is actually orthogonal to any multiple of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(c) Geometrically, what does H look like?

☞ HINT: Use your finger to represent $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What "shape" is created by all the vectors that are orthogonal to your finger?

(d) How must x_1, x_2, x_3 be related so that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in H?

7. Try repeating the previous problem, but now let H consist of all vectors in $\mathbb{R}^3$ that are orthogonal to *both* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

☞ Give TWO descriptions: geometric and as a span.