MATH 220

23 April 2025

Clarification from last time:

If A is an $n\times n$ matrix, then the characteristic polynomial of A has degree n: its highest-exponent term is λ^n

$$\det(A-\lambda I) = \det\begin{bmatrix} ?-\lambda & & & ? \\ & ?-\lambda & & \\ & & \ddots & \\ ? & & & ?-\lambda \end{bmatrix}$$

n copies of λ on the diagonal

This means that A has at most n distinct eigenvalues

More specifically, the algebraic multiplicities of all eigenvalues of A add up to at most n.

it could be less!

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2+1 = 0$$

no (real) eigenvalues

Chapter 8

DOT PRODUCT:

A calculation involving two vectors in $\mathbb{R}^n$

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then $\vec{u}\cdot\vec{v}$ is a number, computed

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Linear Algebra – Day 30

MATH 220

For Problems 1 and 2, let $\mathbf{u} = \begin{bmatrix}3\\4\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}-1\\5\end{bmatrix}$.

1. Compute $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{u}$. How do your results compare?

$\vec{u}\cdot\vec{v} = \begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}-1\\5\end{bmatrix} = 3(-1)+4(5) = 17$

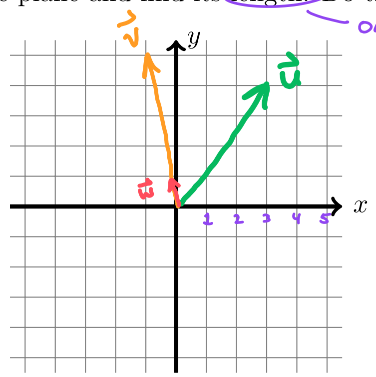
$\vec{v}\cdot\vec{u} = \begin{bmatrix}-1\\5\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix} = (-1)(3)+5(4) = 17$

dot product is commutative

2. (a) Plot vector $\mathbf{u}$ on a coordinate plane and find its length. Do the same for $\mathbf{v}$.

or "norm"

☞ Notation: The length of a vector $\mathbf{u}$ is denoted $||\mathbf{u}||$.



$\|\vec{u}\| = \sqrt{3^2+4^2} = 5$

$\|\vec{v}\| = \sqrt{(-1)^2+5^2} = \sqrt{26}$

(b) Compute the dot product $\mathbf{u}\cdot\mathbf{u}$.

$\vec{u}\cdot\vec{u} = 3\cdot3+4\cdot4 = 25$

(c) How are your answers in parts (a) and (b) related?

$\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$ for all vectors $\vec{u}$

(d) Now draw a random vector $\mathbf{x} = \begin{bmatrix}a\\b\end{bmatrix}$. Can you express the length of $\mathbf{x}$ using the dot product?

$\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$

$\vec{w}$ is a unit vector since its length is 1.

3. Find the vector $\mathbf{w}$ with the following properties:

- the length of $\mathbf{w}$ is 1, and
- $\mathbf{w}$ is in the same direction as $\mathbf{v}$.

$\vec{w} = \frac{1}{\sqrt{26}}\vec{v} = \begin{bmatrix}\frac{-1}{\sqrt{26}}\\ \frac{5}{\sqrt{26}}\end{bmatrix}$

length of $\vec{w}$: $\|\vec{w}\| = \sqrt{\left(\frac{-1}{\sqrt{26}}\right)^2+\left(\frac{5}{\sqrt{26}}\right)^2}$

$= \sqrt{\frac{1}{26}+\frac{25}{26}}$

$= \sqrt{\frac{26}{26}} = \sqrt{1} = 1$

☞ HINT: If a vector had a length of 3 and you wanted to make it have length 1 but not change the direction, what would you do to it?

4. Leo: I just love these two matrices: $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$ and $B = \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$

Cleo: Let's multiply A and B together!

Leo: You're way too enthusiastic about this. But OK. Here it is:

$$AB = \begin{bmatrix}a+2d+3g & b+2e+3h & c+2f+3i\\ 4a+5d+6g & 4b+5e+6h & 4c+5f+6i\\ 7a+8d+9g & 7b+8e+9h & 7c+8f+9i\end{bmatrix}$$

Cleo: OH WOW...there are dot products *everywhere*!

Group chat: Discuss Cleo's observation. In general, how can dot products be used to find the entry in the ith row and jth column of a matrix product AB?

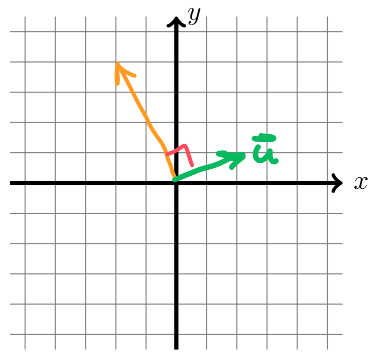
The entry in row i, column j of AB is

(row i of A) $\cdot$ (column j of B)

dot product

5. Now let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

(a) Plot vectors $\mathbf{u}$ and $\mathbf{v}$ and verify geometrically that they are perpendicular to each other.



(b) Compute the dot product $\mathbf{u} \cdot \mathbf{v}$.

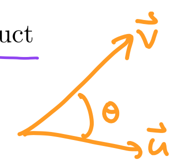
$\vec{u} \cdot \vec{v} = \vec{0}$

(c) Plot a few more vectors that are perpendicular to $\mathbf{u}$. Geometrically, what does the collection of *all* vectors that are perpendicular to $\mathbf{u}$ look like?

(d) Suppose $\mathbf{w}$ is any vector that is perpendicular to $\mathbf{u}$. What can you say about the dot product $\mathbf{u} \cdot \mathbf{w}$? Explain your reasoning.

Must be zero!

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$



6. Let H be the collection of all vectors $\mathbf{x}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \mathbf{x} = 0$.

That is, H is all the vectors that are orthogonal (perpendicular) to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Verify that the vector $\begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$ is in H.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix} = 0$

(b) *True or False*: $\begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}$ is actually orthogonal to any multiple of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

perpendicular

(c) Geometrically, what does H look like? plane through the origin

HINT: Use your finger to represent $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What "shape" is created by all the vectors that are orthogonal to your finger?

(d) How must x_1, x_2, x_3 be related so that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in H?

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \vec{x} = 1x_1 + 2x_2 + 3x_3 = 0$

7. Try repeating the previous problem, but now let H consist of all vectors in $\mathbb{R}^3$ that are orthogonal to *both* $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

Give TWO descriptions: geometric and as a span.

See next page!

Vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is perpendicular to both $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

if and only if $\vec{u} \cdot \vec{x} = x_1 + 2x_2 + x_3 = 0$

and $\vec{v} \cdot \vec{x} = 4x_1 + 5x_2 + 6x_3 = 0$.

That is, we need to solve the system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = 0 \end{cases}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$

So we need $x_1 = x_3$ and $x_2 = -2x_3$,

with x_3 free.

In other words, $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$.

So $H = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$, which is a line in $\mathbb{R}^3$.