MATH 220
21 April 2025

LAST TIME: If $A = PBP^{-1}$, then $A^k = PB^kP^{-1}$!

We want: $A = PDP^{-1}$, where D is diagonal,
since computing D^k is super easy!

If $A = PDP^{-1}$ for some diagonal matrix D and invertible matrix P, then A is diagonalizable.

QUESTIONS: Are all matrices diagonalizable?
How can we find the matrices P and D?

If $A = PDP^{-1}$, then we can write $AP = PD$. $\quad AP = PD\cancel{P^{-1}P}$

What would it take in order to get $AP = PD$?

Suppose: $P = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix}$ and $D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$.

columns

columns of P are eigenvectors of A

d_i are eigenvalues of A

$$AP = PD$$

$$A\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

$$\begin{bmatrix} A\vec{u}_1 & A\vec{u}_2 & \cdots & A\vec{u}_n \end{bmatrix} = \begin{bmatrix} d_1\vec{u}_1 & d_2\vec{u}_2 & \cdots & d_n\vec{u}_n \end{bmatrix}$$

columns! $\qquad$ columns

compare the columns!

For any column i, $A\vec{u}_i = d_i\vec{u}_i$.

So the columns of P **must** be eigenvectors of A with eigenvalues $d_1, d_2, \dots, d_n$.

P has to be invertible, so all of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ must be linearly independent

There are only n eigenvalues (with alg. multiplicity), so we need n linearly indep. eigenvectors, which means:

THEOREM: A is diagonalizable exactly when $alg(\lambda) = geo(\lambda)$ for **every** eigenvalue λ. If this happens, then the columns of P are made from the bases of each E_λ, and the entries of D are the eigenvalues of A.

Linear Algebra – Day 29

MATH 220

For this entire page: $M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

1. Review: What are the eigenvalues of M and of N?

Answer: M is triangular, so the eigenvalues are on the diagonal: $\lambda = 3, 3, 4$

N is also triangular, so the eigenvalues are on the diagonal: $\lambda = 3, 3, 4$

For each matrix, we have $\text{alg}(3) = 2$ and $\text{alg}(4) = 1$.

2. For matrix M: Recall that last time, we found $\text{geo}(3) = 2$ and $\text{geo}(4) = 1$.

Someone at the table: Use Mathematica to compute `Eigensystem[ M ]` to find a basis for E_3 and a basis for E_4.

☜ You will have to input the matrix M first.

M: E_4 has basis $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. E_3 has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

check: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ ✓

3. For matrix N: Last time, we found $\text{geo}(3) = 1$ and $\text{geo}(4) = 1$.

Use Mathematica to find a basis for E_3 and a basis for E_4.

☜ Or do it by hand if you've gotten fast with it!

N: E_4 has a basis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, E_3 has basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

STOP *and wait for further instructions*

4. Erez: Oh wow! So we can conclude from our work above that M is diagonalizable and that N is *not* diagonalizable. That's a shame about N.

Group chat: Why? Also, find a P and D so that $M = PDP^{-1}$:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

OR: $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 4 & & \\ & 3 & \\ & & 3 \end{bmatrix}$

5. Ava: Simon, look at this matrix!

$$C = \begin{bmatrix} 1 & 1 & 5 & 8 & 4 & 0 & 6 & 2 & 4 \\ 0 & 2 & \pi & 7 & 1 & 8 & 2 & 4 & 9 \\ 0 & 0 & 3 & 3 & 4 & 8 & 2 & 3 & 6 \\ 0 & 0 & 0 & 4 & \sqrt{5} & 8 & 2 & 8 & 9 \\ 0 & 0 & 0 & 0 & 5 & 8 & 2 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 2\pi & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Simon: Wow, C is 9×9, so it is going to take forever to find all the eigenspaces.

Ava: But we already know that C is definitely diagonalizable! Cool!

Group chat: How did Ava know that C is diagonalizable without having to do a single computation? C is upper triangular with distinct entries on its diagonal.

So each diagonal entry is an eigenvalue with algebraic and geometric multiplicity 1.

Group chat: If $C = PDP^{-1}$ where D is diagonal, what could the matrix D equal?

D has the numbers 1, 2, 3, 4, 5, 6, 7, 8, 0, in some order, on its diagonal.

6. Suppose A is a (random) 6×6 matrix whose eigenvalues are $\lambda = 2, 2, 3, 3, 3, 4$. With you group, come up with a strategy that will allow you to determine whether or not A is diagonalizable as quickly as possible.

Row-reduce $A-2I$ and $A-3I$, and count the number of non-pivot columns in each. For A to be diagonalizable, we need $\dim(E_2) = 2$ and $\dim(E_3) = 3$.

7. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Each task below requires you to solve the problem posed above it.

(a) What is the characteristic polynomial of A?

$\det(A-\lambda I) = (1-\lambda)(4-\lambda)+2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

(b) What are the eigenvalues of A?

$\lambda = 2, \quad \lambda = 3$

(c) For each eigenvalue λ, find a basis for E_λ.

basis for E_2 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, basis for E_3 is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(d) Determine if A is diagonalizable or not (give a reason).

Yes: for each eigenvalue λ, $\text{alg}(\lambda) = \text{geo}(\lambda) = 1$.

(e) Find a P and diagonal D for which $A = PDP^{-1}$.

$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(f) Find a formula for A^m.

$A^m = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^m & 0 \\ 0 & 3^m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

8. In part (b) of previous problem, a former student made an algebra mistake and ended up saying that $\lambda = 1$ was an eigenvalue of A even though $\lambda = 1$ is actually not an eigenvalue of A. You might think that this poor, unfortunate student is now doomed to get all the rest of the problems wrong.

Group chat: What will happen in part (c) that should alert the student that they did something wrong?

They will find $\dim(E_1) = 0$, which indicates that there is no nonzero vector $\vec{x}$ such that $A\vec{x} = 1\vec{x}$, so $\lambda = 1$ is not an eigenvalue of A.