MATH 220
18 April 2025

RECALL: $$\boxed{A\vec{x} = \lambda\vec{x}}$$

- A: $n \times n$ matrix
- $\vec{x}$: nonzero vector — eigenvector
- λ: scalar — eigenvalue

GOAL: Find the eigenvalues of matrix A.

Statement: λ is an eigenvalue of A.

Rephrase: $A\vec{x} = \lambda\vec{x}$ for a nonzero vector $\vec{x}$.

Rephrase: There is a nonzero vector $\vec{x}$ such that $(A-\lambda I)\vec{x} = \vec{0}$.

Rephrase: $E_\lambda = \text{null}(A-\lambda I)$ contains at least one nonzero vector $\vec{x}$.

Rephrase: $(A-\lambda I)$ is not invertible.

Rephrase: $\det(A-\lambda I) = 0$.

Solution: $\underbrace{\text{Calculate } \det(A-\lambda I), \text{ set equal to } 0}_{\text{characteristic equation}}$, and solve for λ.

NEW STATEMENT FOR THE UNIFYING THM:

$\lambda = 0$ is an eigenvalue of A

$\Updownarrow$ equivalent to

$\det(A) = 0$

$\Updownarrow$ equivalent to

A is <u>not</u> invertible

Linear Algebra – Day 28

MATH 220

1. **Milo:** I wonder what it means for $\lambda = 0$ to be an eigenvalue of A.

☞ Read this aloud at your table!

Delphine: Milo, what it means for λ to be an eigenvalue is written on the screen!

Milo: Oh, right! So let's read what's on the screen and pretend $\lambda = 0$.

Delphine: Wow, something amazing happened!

Group Chat: What amazing thing happens if $\lambda = 0$ is an eigenvalue of A?

$A\vec{x} = \lambda \vec{x}$ if $\lambda = 0$, then $A\vec{x} = \vec{0}$

$\det(A - \lambda I) = \det(A) = 0$

2. Consider the upper-triangular matrices $M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) What are the roots, with multiplicities, of $\det(M - \lambda I)$? How about for $\det(N - \lambda I)$?

$\det(M - \lambda I) = (3-\lambda)^2 (4-\lambda)$

$\det(N - \lambda I) = (3-\lambda)^2 (4-\lambda)$

In both cases, roots: $\lambda = 3$, multiplicity 2

$\lambda = 4$, multiplicity 1

(b) For matrix M, what are $\dim(E_3)$ and $\dim(E_4)$?

$\dim(E_3) = \text{null}(M - 3I) = \text{null} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 2$

$\dim(E_4) = \text{null}(M - 4I) = \text{null} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1$

(c) For matrix N, what are $\dim(E_3)$ and $\dim(E_4)$?

$\dim(E_3) = \text{null}(N - 3I) = \text{null} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

$\dim(E_4) = \text{null}(N - 4I) = \text{null} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1$

STOP *and wait for further instructions*

(d) For M: alg(3) = 2 geo(3) = 2 alg(4) = 1 geo(4) = 1

(e) For N: alg(3) = 2 geo(3) = 1 alg(4) = 1 geo(4) = 1

IMPORTANT: $\text{alg}(\lambda) \geq \text{geo}(\lambda) \geq 1$

3. **Maura:** Wow! This means that if λ only occurs once as a root of the characteristic polynomial of A, we *already* know that λ will produce *only one* dimension worth of eigenvectors. We would not even have to calculate the eigenvectors!

Group Chat: What is Maura talking about?

If $\text{alg}(\lambda) = 1$, then $1 \geq \text{geo}(\lambda) \geq 1$.

Then $\text{geo}(\lambda) = 1$ also.

ORANGE items were written after class

4. (a) Write down any 2×2 diagonal matrix A and any 3×3 diagonal matrix B.

☞ To make it interesting, use different diagonal entries, none equal to 0 or 1.

(b) Find A^2 and B^2.

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

(c) Find A^3 and B^3.

$$B^2 = \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 5^2 & 0 \\ 0 & 0 & (-3)^2 \end{bmatrix}$$

(d) Do you have a guess as to what A^k and B^k equal?

$$B^k = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 5^k & 0 \\ 0 & 0 & (-3)^k \end{bmatrix}$$

(e) If $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$, then what is D^k?

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

(f) **Theorem:** If D is the previous diagonal matrix, then the eigenvalues of D^k are

$d_1^k, d_2^k, \ldots, d_n^k$.

5. Suppose A is an $n \times n$ matrix and $A = PBP^{-1}$ for some $n \times n$ matrices P and B.

(a) Find A^2 and A^3 in terms of P and B.

$$A^2 = PB\underbrace{P^{-1} \cdot P}_{\text{identity matrix}}BP^{-1} = PB \cdot BP^{-1} = PB^2P^{-1}$$

$P^{-1}P = I$

$$A^3 = A^2A = PB^2\underline{P^{-1}P}BP^{-1}$$

$$A^3 = PB^3P^{-1}$$

(b) What do you think A^k equals (in terms of P and B)?

$$A^k = PB^kP^{-1}$$

6. Ava: Don't ask me why, but I really want to know A^{63} for $A = \begin{bmatrix} 1 & 3 & -3 & -3 \\ 3 & -3 & 3 & -1 \\ -3 & 3 & 1 & -3 \\ -3 & -1 & -3 & -3 \end{bmatrix}$.

Jonah: Huh? Why 63? OK, well you can easily find A^{63} using this information:

$$\begin{bmatrix} 1 & 3 & -3 & -3 \\ 3 & -3 & 3 & -1 \\ -3 & 3 & 1 & -3 \\ -3 & -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1}$$

Group chat: Why does Jonah think Ava can calculate A^{63} easily?

☞ Hint: Jonah's information looks like $A = PBP^{-1}$.

B^{63} is easy to calculate, and after that only two matrix multiplications are necessary to compute A^{63}.

DEFINITIONS NOT IN THE BOOK

$(\lambda-3)^{2}$

The number of times λ is a root of the characteristic polynomial is the **algebraic multiplicity** of λ, denoted $\underline{alg(\lambda)}$.

The dimension of E_λ is the **geometric multiplicity** of λ, denoted $\underline{geo(\lambda)}$.