MATH 220
16 April 2025

RECALL: If $\vec{x} \neq 0$ and $\boxed{A\vec{x} = \lambda\vec{x}}$, then $\vec{x}$ is an eigenvector of A corresponding to eigenvalue λ.

Eigenvector $\vec{x}$ must be a solution to $(A-\lambda I)\vec{x} = \vec{0}$.

Example from last time: $A = \begin{bmatrix} 4 & -2 & 5 \\ 1 & 1 & 5 \\ 1 & -2 & 8 \end{bmatrix}$

Simon told us that $\lambda=3$ is an eigenvalue of A.
What are the eigenvectors?

Consider $(A-3I)\vec{x} = \vec{0}$ $\qquad A-3I = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 5 \\ 1 & -2 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 - 2x_2 + 5x_3 = 0$

so: $x_1 = 2x_2 - 5x_3$

null space: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

$E_3 = \text{null}(A-3I) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right)$ $\qquad \dim(E_3) = 2$

every vector in this span is an eigenvector

Is $\lambda=2$ an eigenvalue?

$(A-2I) = \begin{bmatrix} 2 & -2 & 5 \\ 1 & -1 & 5 \\ 1 & -2 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So: $E_2 = \text{null}(A-2I) = \{\vec{0}\}$, and $\lambda=2$ is <u>not</u> an eigenvalue of A

$\dim(E_2) = 0$, meaning <u>no</u> eigenvectors

Definition: $E_\lambda = \text{null}(A-\lambda I)$ is the **eigenspace of A corresponding to λ**

If λ is an eigenvalue, then $\dim(E_\lambda) > 0$.

If λ is not an eigenvalue, then $\dim(E_\lambda) = 0$

How do we find the eigenvalues of a matrix?

example: $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$ $\qquad (A-\lambda I)\vec{x} = \vec{0}$

Are there nontrivial solutions?

rephrase: Does $\text{rref}(A-\lambda I)$ have a free variable?

rephrase: Is $\det(A-\lambda I) = 0$?

This is how we find eigenvalues!

solve

$$0 = \det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & -2 \\ 1 & -2-\lambda \end{bmatrix} = (1-\lambda)(-2-\lambda) - (1)(-2)$$

$$-2-\lambda+2\cancel{\lambda}+\lambda^2 \quad +2$$

$$0 = \cancel{-2}+\lambda+\lambda^2\cancel{+2}$$

$$0 = \lambda+\lambda^2$$

$$0 = \lambda(1+\lambda)$$

eigenvalues!

$$\boxed{\text{So } \lambda = 0 \text{ or } \lambda = -1}$$

Cool thing: $\det(A-\lambda I)$ is <u>always</u> a polynomial in λ

characteristic polynomial of A

The roots (zeros) of the characteristic polynomial are the eigenvalues of A.

Linear Algebra – Day 27

MATH 220

1. (a) Find the characteristic polynomial of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

$\det(A-\lambda I) = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda)-(-1) = \lambda^2-4\lambda+3+1$
$= \lambda^2-4\lambda+4$

(b) What are the eigenvalues of A?

$0 = \lambda^2-4\lambda+4 = (\lambda-2)^2$ so $\lambda = 2$

(c) For one of the λ you found in part (a), find E_λ. (Write it as a span.)

$A-2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

2. Consider the matrix $C = \begin{bmatrix} 5 & 17 & \pi & 312 \\ 0 & -2 & -74 & 91 \\ 0 & 0 & 3 & 803 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

(a) QUICK! Find the characteristic polynomial of C.

☞ Don't panic. Just write $C - \lambda I$ and go from there.

$(5-\lambda)(-2-\lambda)(3-\lambda)(3-\lambda)$

Class ended with problem 2.

(b) When Jason looked at matrix C, he *immediately* said

"Boom! The eigenvalues of this matrix are 5, −2, and 3."

How did Jason know this so quickly?

These are the numbers on the diagonal, and all entries below the diagonal are zero.

Solutions to 3 and 4 are given here.

(c) **Theorem:** If A is a triangular matrix (*either* upper or lower), then you can find the eigenvalues of A by looking at the entries on the diagonal.

3. (a) Suppose you know that A is an invertible $n \times n$ matrix and $A\mathbf{x} = 3\mathbf{x}$. What is $A^{-1}\mathbf{x}$?

☞ HINT: If A^{-1} undoes A and A triples $\mathbf{x}$, what should A^{-1} do to $\mathbf{x}$?

A^{-1} should multiply $\vec{x}$ by $\frac{1}{3}$, so $A^{-1}\vec{x} = \frac{1}{3}\vec{x}$.

(b) Suppose A is a 3×3 matrix whose eigenvalues are $1, 2, 3$. What are the eigenvalues of A^{-1}?

$1, \frac{1}{2}$, and $\frac{1}{3}$

(c) **Conjecture**: If A is an $n \times n$ invertible matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$.

(d) **Conjecture**: A is invertible if and only if zero is not an eigenvalue of A.

☞ Say something about its eigenvalues.

4. (a) Find the characteristic polynomial and eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = (-\lambda)(-\lambda) - (-1)(1) = \lambda^2 + 1$$

(b) What are the eigenvectors (in $\mathbb{R}^2$) for A?

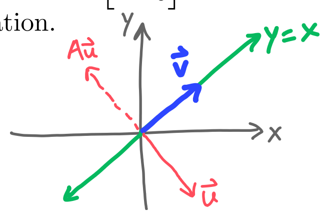
Since $\lambda^2+1=0$ has no (real) solutions, A has no (real) eigenvalues!

Since A has no (real) eigenvalues, it has no (real) eigenvectors.

(c) Recall that A is the matrix that rotates vectors in $\mathbb{R}^2$ counterclockwise by 90 degrees ($\pi/2$ radians). Given that, why do the answers to (a) and (b) make sense?

Rotation of $\mathbb{R}^2$ by 90 degrees changes the direction of every (nonzero) vector. So there is no vector that remains on the same line when multiplied by A.

(d) Use the fact that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is reflection in the line $y = x$ to find the eigenvalues of A without calculation.



$A\vec{v} = \vec{v}$, so 1 is an eigenvalue.

$A\vec{u} = -\vec{u}$, so -1 is an eigenvalue.

Use geometry! What vectors remain multiples of themselves? What multiple?

We will do this on Friday.

5. (a) What is the characteristic polynomial of $M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$? What are the eigenvalues?

(b) What are the dimensions of the eigenspaces of M?

(c) What is the characteristic polynomial of $N = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$? What are the eigenvalues?

(d) What are the dimensions of the eigenspaces of N?

(e) How are M and N the same? How are they different? How are their eigenvalues the same? How are their eignespaces different?