Linear Algebra – Day 12

MATH 220

1. Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 & -4 \\ 1 & 0 & 3 & -2 \\ 2 & 4 & 8 & 1 \end{bmatrix}$$

(a) Rephrase the following question in as many ways as you possibly can:

Is every vector $\mathbf{b}$ *in* $\mathbb{R}^3$ *in the range of* T*?*

(b) What is the answer? Is every vector $\mathbf{b}$ in $\mathbb{R}^3$ in the range of T?

2. Erez: Hey Cleo, it is *amazing* that every linear transformation ON EARTH is really just matrix multiplication.

Cleo: I don't believe it! In class last time, we had a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ x_2 + x_3 \end{bmatrix}$.

Erez: That is an OK way to state it, but there is a 2×3 matrix A that will do the same thing!

Group chat: Why does A have to be 2×3? What is the matrix A so that $T(\mathbf{x})$ is really the same as $A\mathbf{x}$?

3. For a few different $\mathbf{x}$ vectors of your choice, calculate $T(\mathbf{x}) = A\mathbf{x}$ when $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What effect does T have on $\mathbf{x}$?

☞ Why do you think this matrix is called the "identity matrix"?

4. For a few different $\mathbf{x}$ vectors of your choice, calculate $T(\mathbf{x}) = A\mathbf{x}$ when $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. What effect does T have on $\mathbf{x}$? What is T doing to the drawing of $\mathbf{x}$ (*i.e. geometrically*)?

☞ Try drawing the "before" and "after."

5. **Josie:** WOW, Jonah! We can use our new-found knowledge to do cool geometric things!

Jonah: What do you mean, Josie?

Josie: I want to find a matrix that takes vectors in $\mathbb{R}^2$ and rotates them clockwise by $\frac{\pi}{2}$ radians (90 degrees).

Jonah: Well, rotations *are* linear transformations, so there must be a matrix!

Josie: Yes! We only need to know what the transformation does to $\mathbf{e}_1$ and $\mathbf{e}_2$ in order to find the matrix!

Group task:

(a) Make sense of the conversation so far.

(b) What is the result when $\mathbf{e}_1$ is rotated clockwise by $\frac{\pi}{2}$ radians?

(c) What is the result when $\mathbf{e}_2$ is rotated clockwise by $\frac{\pi}{2}$ radians?

(d) What matrix performs *clockwise rotation by* $\frac{\pi}{2}$ *radians*?

6. *Spicy:* Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors in $\mathbb{R}^2$ counterclockwise by $\frac{\pi}{4}$ radians. Find the matrix A that performs T.

☞ Oooooh, a little trigonometry.

7. Suppose $S : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \text{reflection of } \begin{bmatrix} x \\ y \end{bmatrix} \text{ across the } y\text{-axis}$$

(a) On coordinate axes, draw $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and its reflection about the y-axis.

(b) What vector is $S\left(\begin{bmatrix} 5 \\ 3 \end{bmatrix}\right)$?

(c) Find the matrix B such that $S(\mathbf{x}) = B\mathbf{x}$.

☞ Hint: Use $\mathbf{e}_1, \mathbf{e}_2$ somehow.

(d) Is S a linear transformation? Explain how you know.