MATH 220
14 February 2025

LAST TIME

Linear system $\longrightarrow$ Augmented matrix

Elementary row operations: performed to the matrix, don't affect solutions to the system

Echelon form: nonzero entries in each row (pivots) move down and right

WARMUP: If $*$ represents a nonzero number, how many solutions correspond to each matrix?

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right] \qquad \left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \end{array}\right] \qquad \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right]$$

1 solution

free variable

infinitely many solutions

no solutions

EXAMPLE.

$R_1 - R_3$

$R_3 - R_1$

$$\left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 2 & 0 & 3 & 4 & 3 \end{array}\right] \xrightarrow{-R_1+R_3 \to R_3} \left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 3 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{cccc|c} 2 & 0 & 2 & 4 & 2 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

ECHELON FORM!

$$\xrightarrow[-2R_3+R_1 \to R_1]{-R_3+R_2 \to R_2} \left[\begin{array}{cccc|c} 2 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

REDUCED (ROW) ECHELON FORM — RREF

$$\begin{cases} x_1 + 2x_4 = 0 \\ x_2 + 3x_4 = 0 \\ x_3 = 1 \end{cases}$$

SOLUTION:

$$\begin{cases} x_1 = -2s \\ x_2 = -3s \\ x_3 = 1 \\ x = s \end{cases}$$

Linear Algebra – Day 4

MATH 220

1. Milo: I have a matrix A that I need to put in reduced row echelon form.

Ava: I can do that! In fact, I can do it without using any fractions!

$$A = \left[\begin{array}{rrr|r} 2 & -2 & 6 & 4 \\ 6 & -3 & 3 & 0 \\ -1 & 3 & -1 & 2 \end{array}\right]$$

Group chat: Is Ava correct? Can this matrix be put in reduced row echelon form (RREF) without ever using a fraction!

$$\left[\begin{array}{rrr|r} 2 & -2 & 6 & 4 \\ 6 & -3 & 3 & 0 \\ -1 & 3 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & 1 & -2 \\ 2 & -2 & 6 & 4 \\ 6 & -3 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & 1 & -2 \\ 0 & 4 & 4 & 8 \\ 0 & 15 & -3 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -3 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 5 & -1 & 4 \end{array}\right]$$

NOTE: I copied a number incorrectly in class. Corrections are in red.

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 4 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -6 & -6 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 4 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Milo: Fantastic! Now I can read off the solution to the corresponding system!

Group chat: What is the solution?

$$x_1 = 0$$
$$x_2 = 1$$
$$x_3 = 1$$

2. Now put each of the following matrices in RREF.

A few more examples:

$$B = \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 2 & 1 & 3 & 1 \\ -1 & -5 & 3 & 2 \end{array}\right] \qquad C = \left[\begin{array}{rrr|r} 0 & 3 & -5 & -4 \\ 1 & -1 & 3 & 2 \\ -1/3 & 1/3 & -1 & -2/3 \end{array}\right]$$

B: $$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 2 & 1 & 3 & 1 \\ -1 & -5 & 3 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 3 & -3 & -5 \\ 0 & -6 & 6 & 5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & -\frac{5}{3} \\ 0 & -6 & 6 & 5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & -1 & -\frac{5}{3} \\ 0 & 0 & 0 & -5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 2 & \frac{4}{3} \\ 0 & 1 & -1 & -\frac{5}{3} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution!

C: $$\left[\begin{array}{rrr|r} 0 & 3 & -5 & -4 \\ 1 & -1 & 3 & 2 \\ -\frac{1}{3} & \frac{1}{3} & -1 & -\frac{2}{3} \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 2 \\ 0 & 3 & -5 & -4 \\ -1 & 1 & -3 & -2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 2 \\ 0 & 3 & -5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -1 & 3 & 2 \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: $$\begin{cases} x_1 = \frac{2}{3} - \frac{4}{3}s \\ x_2 = \frac{-4}{3} + \frac{5}{3}s \\ x_3 = s \end{cases}$$

3. For each of the augmented matrices A, B, and C on this page, is the corresponding solution set empty, a point, a line, or a plane?

think about this...

4. A coffee merchant sells 3 coffee blends. One bag of each blend contains the following types of coffee beans:

- The House blend is 300g of Colombian and 200g of Guatemalan coffee beans.
- The Special blend is 200g of Colombian, 200g of Kenyan, and 100g of Guatemalan beans.
- The Gourmet blend is 100g of Colombian, 200g of Kenyan, and 200g of Guatemalan beans.

The merchant has on hand 30kg of Colombian, 15kg of Kenyan, and 25kg of Guatemalan beans.

(a) Write a system of equations whose solution tells you how many bags of each blend the merchant should prepare in order to use all the beans.

(b) Set up the augmented matrix associated with your system.

(c) What does the first row in your matrix represent?

(d) What does the first column in your matrix represent?

(e) Row reduce your matrix. What is the solution to the system?

We didn't do this problem in class, but here is the solution.

Let x_1 be the number of bags of house blend, x_2 the number of bags of special blend, and x_3 the number of bags of gourmet blend.

Then:

$$\begin{aligned} 300x_1 + 200x_2 + 100x_3 &= 30000 &\leftarrow \text{grams of Colombian beans} \\ 200x_2 + 200x_3 &= 15000 &\leftarrow \text{grams of Kenyan beans} \\ 200x_1 + 100x_2 + 200x_3 &= 25000 &\leftarrow \text{grams of Guatemalan beans} \end{aligned}$$

In matrix form:

$$\left[\begin{array}{ccc|c} 300 & 200 & 100 & 30000 \\ 0 & 200 & 200 & 15000 \\ 200 & 100 & 200 & 25000 \end{array}\right]$$

Reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 45 \end{array}\right]$$

So the merchant should make 60 bags of house blend, 30 bags of special blend, and 45 bags of gourmet blend.

These are good practice problems!

5. Suppose that the echelon form of an augmented matrix has a pivot position in every column. How many solutions does the associated linear system have? Explain.

None — a pivot in the rightmost column of an augmented matrix implies no solution.

Example:

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{array}\right]$$

6. Suppose you have a homogeneous linear system with more variables than equations. How many solutions will it have? Why?

↳ the right-hand side of each equation is zero

A homogeneous linear system is always consistent, since it has at least the trivial solution. If it has more variables than equations, then it has at least one free variable. Therefore, it has infinitely many solutions.