MATH 126 A/B — 21 Nov. 2025

Exam 3 Practice Problems

1. Let $f(x, y) = e^{-x^2-y^2}$.

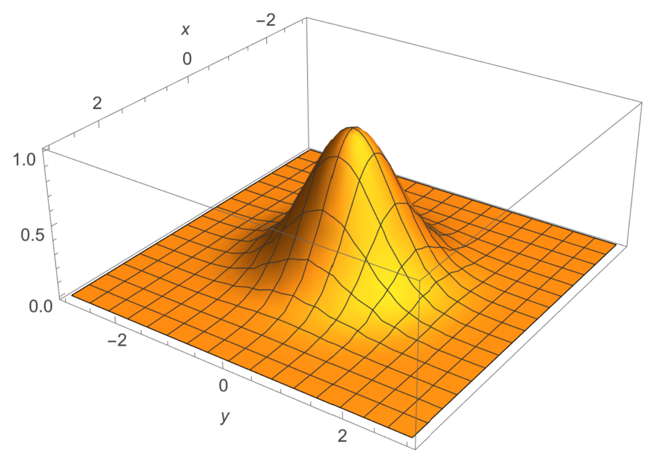
(a) Make a guess for what the graph of this function looks like without using technology. (*Hint*: What is $f(0, 0)$? What happens to $f(x, y)$ as (x, y) move away from the origin?)

(b) Use technology to draw the graph. Was your guess correct?

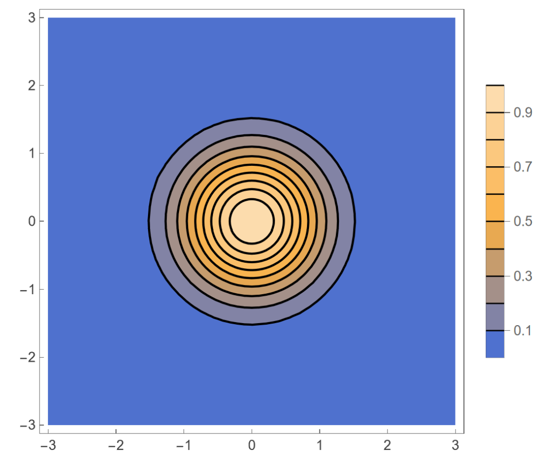
(c) Sketch the contour diagram for this function.

(a) $f(0,0) = 1$, and $f(x,y)$ decreases toward zero as (x,y) moves away from the origin in any direction.

(b) Graph of $f(x,y) = e^{-x^2-y^2}$:

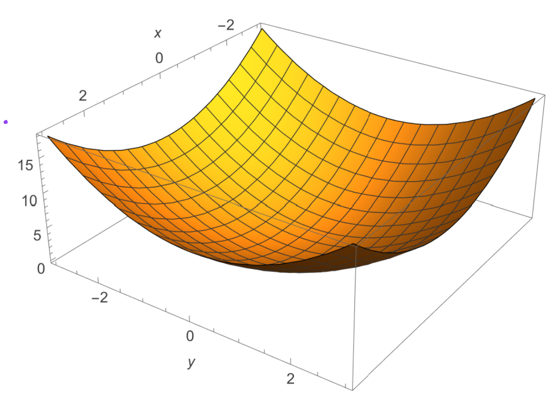


(c) Contour diagram:



2. Explain what is wrong with the following statement: "Since $x^2 + y^2 = 1$ is the equation of a circle, the graph of $f(x, y) = x^2 + y^2$ is a circle."

In 3D, the graph of $f(x,y) = x^2 + y^2$ is a paraboloid (sort of like a bowl). Its level curves are circles, since for any fixed value of z, $z = x^2+y^2$ is a circle.



3. Is it true that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$? Either explain why this is true for all vectors, or give a counterexample that shows it is not true for some vectors.

True! Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, and $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

Then: $\vec{u} \cdot (\vec{v} + \vec{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$.

SAME!

Also: $\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle + \langle u_1, u_2, u_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_1 w_1 + u_1 w_2 + u_1 w_3 = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3).$$

This shows that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

4. Find the equation of the plane through the points $(2, 0, 3)$, $(5, 2, -2)$, and $(1, 4, -1)$.

Let $A = (2,0,3)$, $B = (5,2,-2)$, and $C = (1,4,-1)$.

Then $\vec{AB} = \langle 3, 2, -5 \rangle$ and $\vec{AC} = \langle -1, 4, -4 \rangle$.

A normal vector to the desired plane is $\vec{AB} \times \vec{AC} = \langle 12, 17, 14 \rangle$.

Thus, the equation of the plane is

$$12(x-2) + 17(y-0) + 14(z-3) = 0,$$

which simplifies to $12x + 17y + 14z = 66$.

5. A plane has equation $z = 5x - 2y + 7$.

(a) Find a value of a making the vector $a\mathbf{i} + \mathbf{j} + 0.5\mathbf{k}$ normal (i.e., perpendicular) to the plane.

(b) Find a value of b so that the point $(b + 1, b, b - 1)$ lies in the plane.

(a) The equation of the plane is $5x - 2y - z = -7$,

which tells us that the vector $\vec{n} = \langle 5, -2, -1 \rangle$ is normal to the plane.

We want $\langle a, 1, \frac{1}{2} \rangle$ to be a multiple of $\langle 5, -2, -1 \rangle$,

so choose $a = \frac{-5}{2}$.

(b) If $(x, y, z) = (b+1, b, b-1)$ is a point on the plane, then

$$5(b+1) - 2(b) - (b-1) = -7$$

$$2b + 6 = -7$$

$$2b = -13$$

so $\boxed{b = -\frac{13}{2}}$

6. Consider the plane $x + 3y - 2z = 4$ and the vector $\mathbf{v} = \langle 1, 2, 3 \rangle$.

(a) Find a normal vector to the plane.

(b) What is the angle between $\mathbf{v}$ and the vector you found in part (a)?

(c) What is the angle between $\mathbf{v}$ and the plane?

(a) The normal vector is given by the coefficients of x, y, and z in the equation of the plane. For this plane, the normal vector is

$$\vec{n} = \langle 1, 3, -2 \rangle.$$

(b) Let θ be the angle between $\vec{n}$ and $\vec{v}$. Then $\vec{n} \cdot \vec{v} = |\vec{n}||\vec{v}| \cos(\theta)$.

So: $$\cos(\theta) = \frac{\vec{n} \cdot \vec{v}}{|\vec{n}|\,|\vec{v}|} = \frac{\langle 1,3,-2 \rangle \cdot \langle 1,2,3 \rangle}{\sqrt{1^2+3^2+(-2)^2}\,\sqrt{1^2+2^2+3^2}} = \frac{1+6-6}{\sqrt{14}\,\sqrt{14}} = \frac{1}{14}$$

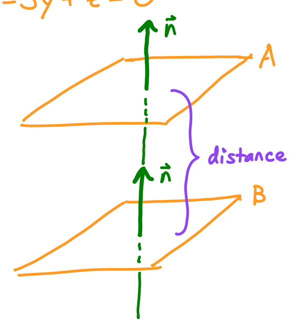
Thus, $\theta = \arccos\left(\frac{1}{14}\right) = 1.499$ radians or 85.9 degrees.

(c) Let α be the angle between $\vec{v}$ and the plane.

Then α and θ are complementary angles, meaning they add up to 90 degrees.

So $\alpha = \frac{\pi}{2} - \theta = 0.0715$ radians or $\alpha = 90 - \theta = 4.1$ degrees.

7. *Spicy:* Find the shortest distance between the planes $\underbrace{2x - 5y + z = 10}_{\text{plane A}}$ and $\underbrace{z = 5y - 2x}_{\text{plane B, also written } 2x - 5y + z = 0}$.



First, note that the vector $\vec{n} = \langle 2, -5, 1 \rangle$ is normal to both planes, so the planes are parallel

The shortest distance between the planes is the distance between the points where a line parallel to $\vec{n}$ intersects the planes.

One such line is given by $x = 2t$, $y = -5t$, $z = t$.

Plane A intersects the line: $2(2t) - 5(-5t) + (t) = 10$

$30t = 10$, so $t = \frac{1}{3}$.

Point of intersection: $\left(\frac{2}{3}, \frac{-5}{3}, \frac{1}{3}\right)$.

Plane B intersects the line: $2(2t) - 5(-5t) + (t) = 0$

$30t = 0$, so $t = 0$.

Point of intersection: $(0,0,0)$.

Thus, the distance between the planes is the distance from $(0,0,0)$ to $\left(\frac{2}{3}, \frac{-5}{3}, \frac{1}{3}\right)$,

which is $\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-5}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{25}{9} + \frac{1}{9}} = \frac{\sqrt{30}}{3} \approx 1.828.$

8. Give an example of a vector $\mathbf{v}$ whose cross product with $\mathbf{u} = \mathbf{i} + \mathbf{j}$ is parallel to $\mathbf{k}$.

One example is $\vec{v} = \vec{i} = \langle 1,0,0\rangle$.

$\vec{i} \times (\vec{i} + \vec{j}) = \langle 1,0,0\rangle \times \langle 1,1,0\rangle = \langle 0,0,1\rangle = k$

Any nonzero vector $\vec{v}$ in the xy-plane will also work.

9. Let $f(x, y) = 3x^2 + 4y^2 - axy$.

(a) Find the values of a such that the graph of f slopes upwards when moving from the point $(1, 2)$ in the positive x-direction.

(b) Find the values of a such that the graph of f slopes upwards when moving from the point $(1, 2)$ in the positive y-direction.

(a) We want to find a such that $f_x(1,2) > 0$.

Since $f_x(x,y) = 6x - ay$, $f_x(1,2) = 6 - 2a$.

We want $6 - 2a > 0$, so we need $6 > 2a$, or $a < 3$.

(b) Now we want to find a such that $f_y(1,2) > 0$.

Since $f_y(x,y) = 8y - ax$, $f_y(1,2) = 16 - a$.

We want $16 - a > 0$, so we need $a < 16$.

10. Give an example of a nonlinear function $f(x, y)$ such that $f_x(0, 0) = 3$ and $f_y(0, 0) = 4$.

One example is $f(x,y) = x^2 + 3x + 4y$.

Many other examples are possible.

11. Let $h(x, y)$ be a differentiable function such that $h(300, 200) = 50$, $h_x(300, 200) = 8$, and $h_y(300, 200) = -6$. Estimate the value of $h(305, 196)$.

From the given information, the tangent plane (linearization) to $h(x,y)$ at $(300, 200)$ is $L(x,y) = 50 + 8(x-300) - 6(y-200)$.

Then we estimate $h(305, 196)$ as:

$$L(305, 196) = 50 + 8(305-300) - 6(196-200) = 50 + 8(5) - 6(-4) = 114.$$

12. If $z = \underbrace{f(x)} + y\underbrace{g(x)}$, what can you say about z_{yy}? Explain!

Note that f and g depend only on x,

so their derivatives with respect to y are zero.

Partial derivative of z with respect to y: $z_y = g(x)$.

So z_y does not depend on y, and thus $z_{yy} = \frac{\partial}{\partial y}(g(x)) = 0$.

Thus, $z_{yy} = 0$.

13. Let $f(x, y) = x^2 + 2y^2$. Find the directional derivative at the point $(0, \pi/4)$ in the direction of $\mathbf{i} + \mathbf{j}$.

First, find the gradient: $\nabla f(x,y) = \langle 2x, 4y \rangle$. So $\nabla f(0, \frac{\pi}{4}) = \langle 0, \pi \rangle$.

We also need a unit vector in the direction of $\vec{i} + \vec{j}$:

$$\vec{u} = \frac{\vec{i}+\vec{j}}{|\vec{i}+\vec{j}|} = \frac{\langle 1,1 \rangle}{|\langle 1,1 \rangle|} = \frac{1}{\sqrt{2}}\langle 1,1 \rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Then the directional derivative is:

$$D_{\vec{u}} f(0, \tfrac{\pi}{4}) = \nabla f(0, \tfrac{\pi}{4}) \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \langle 0, \pi \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{\pi}{\sqrt{2}}.$$

14. Let $g(x, y) = ye^{-x^2} + 2y$. Find the gradient of g and the maximum value of the directional derivative of g at the point $(0, 4)$.

First, the gradient: $\nabla g(x,y) = \langle -2xye^{-x^2}, e^{-x^2} + 2 \rangle$

The gradient at $(0,4)$: $\nabla g(0,4) = \langle -2(0)(4)e^{-0^2}, e^{-0^2} + 2 \rangle = \langle 0, 3 \rangle$

The maximum value of the directional derivative at $(0,4)$ is the magnitude of the gradient at $(0,4)$, which is $|\nabla g(0,4)| = |\langle 0,3 \rangle| = 3$.