MATH 126 A/B — 19 Nov. 2025

RECALL: $f(x,y)$

$\frac{\partial f}{\partial x} = f_x$ is the rate f changes in the direction $\langle 1,0\rangle$.

$\frac{\partial f}{\partial y} = f_y$ is the rate f changes in the direction $\langle 0,1\rangle$.

RATE OF CHANGE IN A GIVEN DIRECTION

Choose a direction, given by a vector $\vec{u}$.

We want to know the rate at which $z=f(x,y)$ changes in that direction.

That is: What is Δz if we move 1 unit in the $\vec{u}$-direction?

Important: Make $\vec{u}$ a unit vector. If $|\vec{u}| \neq 1$, use $\frac{\vec{u}}{|\vec{u}|}$ instead.

Assuming $\vec{u}$ now has length 1, the rate at which f changes in the $\vec{u}$-direction is called the directional derivative of f in the $\vec{u}$-direction and is denoted $D_{\vec{u}} f(x,y)$.

How is it calculated?

$\vec{u} = \langle a,b\rangle$ is a unit vector.

Then: $\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b}$

DEFINITION: The gradient of f is the vector $\langle f_x(x,y), f_y(x,y)\rangle$.

Notation: $\nabla f(x,y)$ or ∇f

So, using this notation: $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

THEOREM: $\nabla f(x,y)$ points in the direction of steepest ascent, and the slope in that direction equals the length of the gradient vector.

Directional Derivatives

1. The wind chill function $w(T, v)$ is on the screen again. As we did last week, focus on the entry corresponding to $T = 10$ and $v = 15$. That is, $w(10, 15) = -7$.

Recall from last week:

- If $\Delta T = 5$, then $\Delta w = 7$. So, an *estimate* for the partial derivative in the T direction is $w_T(15, 20) \approx \frac{7}{5}$.
- If $\Delta v = 5$, then $\Delta w = -2$. So, an *estimate* for the partial derivative in the v direction is $w_v(15, 20) \approx -\frac{2}{5}$.

(a) What is your best guess for ~~the rate that w changes~~ Δw if $\Delta T = 5$ *and* $\Delta v = 5$?

Both variables change!

$$\Delta w = w(15,20) - w(10,15) = -2 - (-7) = 5$$

(b) What is your best guess for ~~the rate that w changes~~ Δw if $\Delta T = 1$ *and* $\Delta v = 1$?

Both variables change!

This is one-fifth the change in both variables as in part (a),

So it seems that $\Delta w = \dfrac{w(15,20) - w(10,15)}{5} = \dfrac{5}{5} = 1$

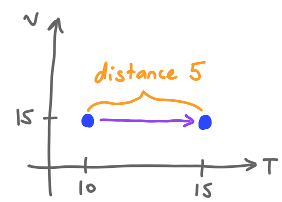
(c) **Milo:** Hey Jade, $w_T(10, 15)$ is really a rate of change!

Jade: It sure is, a rate of change, Milo! That means we can think about it like this:

$$w_T(10, 15) \approx \frac{\Delta w}{\Delta T} = \frac{w(15, 15) - w(10, 15)}{5}$$

$$= \frac{w(15, 15) - w(10, 15)}{\text{the distance between the points } (T, v) = (15, 15) \text{ and } (T, v) = (10, 15)}$$

Group chat: Why is the previous fraction actually equal to $\frac{7}{5}$?



NUMERATOR: $w(15,15) - w(10,15) = 0 - (-7) = 7$

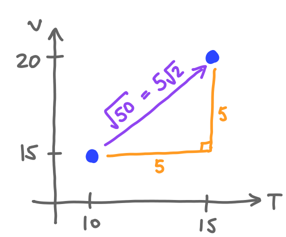
DENOMINATOR: the distance between $(15,15)$ and $(10,15)$ is 5, as shown in the diagram at right

(d) **Milo:** Could we also talk about a rate of change in the direction where $\Delta T = 5$ *and* $\Delta v = 5$?

Jade: We sure can! In this case, our fraction becomes:

$$\text{rate of change} = \frac{w(15, 20) - w(10, 15)}{\text{the distance between the points } (T, v) = (15, 20) \text{ and } (T, v) = (10, 15)}$$

Group chat: What is this new fraction equal to?



NUMERATOR: $w(15,20) - w(10,15) = -2 - (-7) = 5$

DENOMINATOR: the distance between $(15,20)$ and $(10,15)$ is $5\sqrt{2}$, as shown in the diagram at right

Thus, the rate of change is $\dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}} \approx 0.7$

Stop and wait for further instructions. While you wait, discuss: Would it make sense to talk about the rate that w changes for other combinations of ΔT and Δv?

2. What is the rate at which $f(x,y) = 2x^2 + y^2 - 5$ is changing in the direction $\mathbf{u} = \langle 3,4 \rangle$ at the point $(x,y) = (1,3)$?

Is $\vec{u}$ a unit vector? No: $|\vec{u}| = \sqrt{3^2+4^2} = 5$

So rescale: $\frac{\vec{u}}{|\vec{u}|} = \frac{1}{5}\langle 3,4\rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

Use this vector!

$a = \frac{3}{5}$, $b = \frac{4}{5}$

Partial Derivatives:

$f_x = 4x$ $\qquad f_y = 2y$

$f_x(1,3) = 4(1) = 4$ $\qquad f_y(1,3) = 2(3) = 6$

Thus, the rate of change is $D_{\vec{u}} f(1,3) = 4 \cdot \frac{3}{5} + 6 \cdot \frac{4}{5} = 7.2$

3. Milo: WOW! We have a formula for the directional derivative when $\mathbf{u}$ is the unit vector $\langle a, b\rangle$:

$$D_{\mathbf{u}} f(x,y) = f_x(x,y) \cdot a + f_y(x,y) \cdot b.$$

Jade: Milo, look!! $D_{\mathbf{u}} f(x,y)$ is the *dot product* of two vectors!

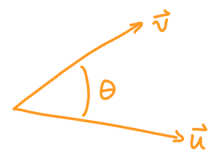
Group chat: Which two vectors is $f_x(x,y) \cdot a + f_y(x,y) \cdot b$ the dot product of?

$$f_x(x,y) \cdot a + f_y(x,y) \cdot b = \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$$

Milo: One of those vectors is $\mathbf{u}$. The other vector must be important, so maybe we should give it a name?

$\langle f_x(x,y), f_y(x,y) \rangle$ is the gradient vector, denoted $\nabla f(x,y)$.

4. Jade: Hey, do you remember the formula $\mathbf{v} \cdot \mathbf{u} = |\mathbf{v}|\,|\mathbf{u}| \cos(\theta)$?



Milo: I sure do! Let's try to apply it to $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$.

Jade: OK, if we substitute the vectors into the formula, we get

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = \underbrace{|\nabla f(x,y)|}_{\text{some number}}\,\underbrace{|\mathbf{u}|}_{1} \underbrace{\cos(\theta)}_{-1 \le \cos(\theta) \le 1}.$$

What good is that?

Milo: Well, we know $|\mathbf{u}| = 1$ and we know $|\nabla f(x,y)|$ is some number. So, $D_{\mathbf{u}} f(x,y)$ is as large as it could possibly be when $\cos(\theta)$ equals 1.

Group chat: What angle θ makes $D_{\mathbf{u}} f(x,y)$ as large as it could possibly be? What does this mean about the vectors $\nabla f(x,y)$ and $\mathbf{u}$? θ = zero, so vectors $\vec{u}$ and $\nabla f(x,y)$ have the same direction

The steepest rate of change is in the direction of the gradient, and this rate of change is the length (magnitude) of the gradient vector.

class ended here

Group chat: What angle θ makes $D_{\mathbf{u}} f(x,y)$ as *small** as it could possibly be? What does this mean about the vectors $\nabla f(x,y)$ and $\mathbf{u}$? $\theta = 180°$, so vectors $\vec{u}$ and $\nabla f(x,y)$ have opposite directions

☞ *Negative numbers are, in fact, smaller than 0.

The rate of steepest descent is in the direction opposite the gradient vector.

5. Let $f(x,y) = xy + 2x^2 - 3y$. On the graph of f, what is the direction of the steepest slope at the point $(2,1)$? What is this steepest slope?

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle y + 4x, x - 3 \rangle$

$\nabla f(2,1) = \langle 1 + 4(2), 2 - 3 \rangle = \langle 9, -1 \rangle$

Steepest slope at $(2,1)$ is in the direction of $\langle 9,-1 \rangle$.

This steepest slope is $|\langle 9,-1\rangle| = \sqrt{9^2 + (-1)^2} = \sqrt{82} \approx 9.055$