MATH 126 A/B — 17 November 2025

Recall from Calc I:

The point-slope equation for a line in 2D is

$$y = y_0 + m(x - x_0)$$

The equation of the line tangent to $y=f(x)$ at $x = x_0$ can be written

$$y = f(x_0) + f'(x_0)(x - x_0)$$

EXAMPLE: If $f(x) = 3x^2 + x$, find the tangent line at $(1,4)$.

First, $f'(x) = 6x + 1$, so $f'(1) = 7$.

Then the tangent line is $y = 4 + 7(x-1)$.

In 3D:

The graph of a function $z = f(x,y)$ has a tangent plane at any point where the first partial derivatives of f exist and are continuous.

The equation of this tangent plane is:

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The tangent plane is also called the linearization of $f(x,y)$ at the point (x_0,y_0) and is written

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The linearization can be used to approximate values of f near (x_0,y_0).

Tangent Planes

1. Group Conjecture: What is the equation for the plane tangent to $z = f(x, y)$ when $(x, y) = (x_0, y_0)$?

☜ Look at the equation for a tangent line on the wall. Then fill in the blanks here.

$$z = \underline{f(x_0,y_0)} + \underbrace{\underline{f_x(x_0,y_0)}}_{\text{slope}}(x - \underline{x_0}) + \underbrace{\underline{f_y(x_0,y_0)}}_{\text{slope}}(y - \underline{y_0})$$

2. Find the equation of the plane that is tangent to $f(x, y) = 2x^2 + y^2 - 3y$ at $(x, y) = (1, 1)$.

First, $f(1,1) = 2(1)^2 + (1)^2 - 3(1) = 2 + 1 - 3 = 0$

Next: $f_x = 4x$ and $f_y = 2y - 3$

$f_x(1,1) = 4(1) = 4$ $\quad f_y(1,1) = 2(1) - 3 = -1$

Tangent plane: $z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$z = 0 + 4(x-1) + (-1)(y-1)$

$$\boxed{z = 4x - y - 3}$$

3. Erez: Hey! The tangent plane is really close to the original function for pairs of (x, y) that are "close" to $(1, 1)$.

Jade: Oh, so we can approximate $f(1.1, 0.9)$ without actually using $f(x)$.

Group task: Help Jade approximate $f(1.1, 0.9)$ without actually using $f(x)$.

Linearization: $L(x,y) = 4x - y - 3$

That is: $f(x,y) \approx L(x,y) = 4x - y - 3$

$f(1.1, 0.9) \approx L(1.1, 0.9) = 4(1.1) - (0.9) - 3$

$= 4.4 - 3.9 = \boxed{0.5}$

Exact value of $f(1.1, 0.9)$:

$f(1.1, 0.9) = 2(1.1)^2 + (0.9)^2 - 3(0.9)$

$= \underline{0.53}$

Close!

4. Use a tangent plane to estimate the value of $g(x, y) = x \sin(x + y)$ at $(x, y) = (0.5, 3)$.

Choose a point of tangency (x_0, y_0) that is <u>near</u> $(0.5, 3)$ and such that $g(x_0, y_0)$ is <u>easy</u> to evaluate.

One choice: $(x_0, y_0) = (0, \pi)$. Then $g(0,\pi) = 0 \sin(0+\pi) = 0$.

Partial derivatives: $g_x = 1 \sin(x+y) + x\cos(x+y)$ so $g_x(0,\pi) = \sin(\pi) + 0\cos(\pi) = 0$

$g_y = x\cos(x+y)$ so $g_y(0,\pi) = 0\cos(\pi) = 0$

Linearization: $L(x,y) = 0 + 0(x-0) + 0(y-\pi)$

then our estimate is $L(0.5, 3) = 0$

5. Find the equation of the plane tangent to $f(x,y) = 3 - 2x + 5y$ at $(x,y) = (2,3)$. Simplify as much as possible.

Does the result surprise you?

$f(2,3) = 3 - 2(2) + 5(3) = 14$

$f_x = -2$ so $f_x(2,3) = -2$

$f_y = 5$ so $f_y(2,3) = 5$

SAME!

Equation of tangent plane:

$z = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3)$

$z = 14 + (-2)(x-2) + 5(y-3)$

$\boxed{z = -2x + 5y + 3}$

If $f(x,y)$ is a plane, then it is its own tangent plane!

6. Find the linearization of $f(x,y) = \sqrt{10 - x^2 - 2y^2}$ at $(2,1)$ and use it to approximate $f(1.9, 1.1)$.

$f(2,1) = \sqrt{10 - (2)^2 - 2(1)^2} = \sqrt{10-4-2} = \sqrt{4} = 2$

$f_x = \frac{1}{2}(10 - x^2 - 2y^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{10-x^2-2y^2}}$ so $f_x(2,1) = \dfrac{-2}{\sqrt{4}} = -1$

$f_y = \frac{1}{2}(10 - x^2 - 2y^2)^{-1/2}(-4y) = \dfrac{-2y}{\sqrt{10-x^2-2y^2}}$ so $f_y(2,1) = \dfrac{-2(1)}{\sqrt{4}} = -1$

Linearization: $L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$= 2 + (-1)(x-2) + (-1)(y-1) = 5 - x - y$

Estimate: $f(1.9, 1.1) \approx L(1.9, 1.1) = 5 - 1.9 - 1.1 = \boxed{2}$

Exact value:
$f(1.9, 1.1) = \sqrt{10 - (1.9)^2 - 2(1.1)^2}$
$= 1.992$

7. Suppose that your friend claims that the equation of the tangent plane to the graph of $f(x,y) = x^3 - y^2$ at the point $(4,5)$ is

$$z = 39 + 3x^2(x-4) - 2y(y-5).$$

(a) Why is this not possibly the equation of a tangent plane?

This is not a linear equation since it includes x^3 and y^2 terms.

(b) What mistake did your friend make?

Forgot to evaluate the partial derivatives f_x and f_y at $(4,5)$.

(c) What is the correct equation of the tangent plane?

$f_x = 3x^2$, so $f_x(4,5) = 48$ and $f_y = -2y$ so $f_y(4,5) = -10$

Tangent plane: $z = 39 + 48(x-4) - 10(y-5)$

8. Find the linearization of $z = y\ln(x)$ at $(1,4,0)$.

$f(x,y) = y\ln(x)$ $\qquad f(1,4) = 0$

$f_x = \frac{y}{x}$ so $f_x(1,4) = \frac{4}{1} = 4$

$f_y = \ln(x)$ so $f_y(1,4) = \ln(1) = 0$

Linearization:

$L(x,y) = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4)$

$= 0 + 4(x-1) + 0(y-4)$

thus, $\boxed{L(x,y) = 4x - 4}$