MATH 126 A/B — 17 October 2025

Informal definition: An infinitely long polynomial is called a **power series**.

Formally: A series that has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where $c_0, c_1, c_2, \ldots$ are constants is called a **power series**.

Example: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ for $\left|\frac{x}{2}\right| < 1$

$-2 < x < 2$

REMEMBER: The ratio test helps you find the values of x that allow a power series to converge.

This gives you an <u>interval</u> of values (could be only a single value or all real numbers).

The radius of this interval is the **radius of convergence**.

Example: $\displaystyle\sum_{n=0}^{\infty} x^n$ converges if $-1 < x < 1$. radius of convergence: 1

Power Series

1. Warm-up:

Geometric series with common ratio $r = x$

(a) What numerical values of x will make the series $\displaystyle\sum_{n=0}^{\infty} x^n$ converge?

☞ Would it be easier for you if we used the letter r instead of the letter x?

$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

converges if $|x| < 1$ or $-1 < x < 1$.

(b) For the values of x you found in (a), what is the *actual sum* of the series?

☞ Hopefully There should be an x in your answer, since the sum depends on what x is.

The sum is $\dfrac{a}{1-r} = \dfrac{1}{1-x}$

(c) What numerical values of x will make the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ converge?

geometric series with common ratio $r = \frac{x}{2}$

Converges if $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$ or $-2 < x < 2$

(d) For the values of x you found in (c), what is the actual sum of the series?

The sum is $\dfrac{a}{1-r} = \dfrac{1}{1-\frac{x}{2}} = \dfrac{2}{2-x}$.

2. Chloe: My personal favorite series is:

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \cdots$$

geometric series
common ratio: $r = 2x$
first term: $a = 1$

Can you help me write this in summation notation, please?

Group task: Help Chloe out.

$\displaystyle\sum_{n=0}^{\infty} (2x)^n$

Simon: Your series converges to $\dfrac{1}{1-2x}$.

Chloe: Thanks, Simon, but I think you forgot one detail.

Group chat: What values of x allow Chloe's series to converge? Is the sum really $\dfrac{1}{1-2x}$?

$|2x| < 1$ or $|x| < \frac{1}{2}$ or $\frac{-1}{2} < x < \frac{1}{2}$

3. Find the sum of this series (in terms of x): $\underbrace{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}_{\text{geometric series with } r = x} = \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$

Which values of x allow this series to converge?

Series converges if $|-x| < 1$ or $|x| < 1$

4. Can you figure out what this series converges to and which values of x allow the series to converge?

$$\underbrace{1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \cdots}_{\text{geometric series with } r = x^2} = \frac{1}{1-x^2}$$

Series converges if $|x^2| < 1$ or $|x| < 1$

or $x^2 < 1$

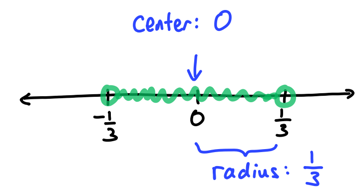
5. Let's try the problem backwards now!

Convergence Interval

(a) Come up with a power series that converges to $\dfrac{1}{1-3x}$.

Try a geometric series with $a=1$, $r=3x$.

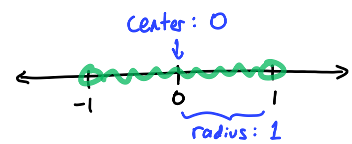
$$\frac{1}{1-3x} = \sum_{n=0}^{\infty}(3x)^n \quad \text{for } |3x|<1 \text{ or } |x|<\tfrac{1}{3}$$



(b) Come up with a power series that converges to $\dfrac{1}{1-x^3}$.

$a=1$, $r=x^3$

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty}(x^3)^n = \sum_{n=0}^{\infty} x^{3n} \quad \text{for } |x^3|<1 \text{ or } |x|<1$$

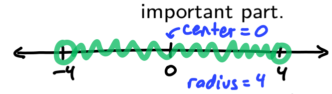


(c) Come up with a power series that converges to $\dfrac{1}{4+x}$.

☞ Use algebra to make this look like $\frac{a}{1-r}$. Getting the "1" in place is the MOST important part.

$$\frac{1\cdot\frac{1}{4}}{(4+x)\frac{1}{4}} = \frac{\frac{1}{4}}{1+\frac{x}{4}} \qquad \text{So } a=\tfrac{1}{4},\ r=\tfrac{-x}{4}$$

$$\frac{1}{4+x} = \sum_{n=0}^{\infty}\frac{1}{4}\left(\frac{-x}{4}\right)^n \quad \text{for } \left|\frac{x}{4}\right|<1 \text{ or } |x|<4$$

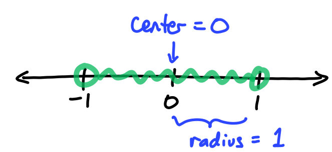


(d) Come up with a power series that converges to $\dfrac{x^2}{1-x}$.

☞ Factor out x^2.

$a=x^2$, $r=x$

$$\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^2(x^n) = \sum_{n=0}^{\infty} x^{n+2} \quad \text{for } |x|<1$$



6. For what numerical values of x does each series converge? What is the radius of convergence for series?

We only did (a) in class, but (b) and (c) are good examples, too!

Ratio Test:

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ $\qquad a_n = \dfrac{x^n}{n!}$

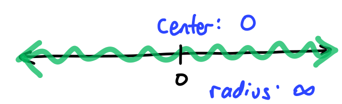
here, x is fixed and $n\to\infty$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{\cancel{(n+1)!}_{\,n+1}}\cdot\frac{\cancel{n!}}{\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

Interval of convergence: $-\infty < x < \infty$

☞ Cool fact: $0! = 1$.

This series converges for all values of x.

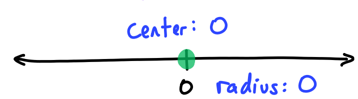


(b) $\displaystyle\sum_{n=0}^{\infty} n!\,x^n$ $\qquad a_n = n!\,x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(n+1)!}^{\,n+1}\ \cancel{x^{n+1}}^{\,x}}{\cancel{n!}\ \cancel{x^n}}\right| = \lim_{n\to\infty}|(n+1)x| = \begin{cases} 0 & \text{if } x=0 \\ \text{DNE } (\infty) & \text{if } x\neq 0\end{cases}$$

Interval of convergence: $x=0$

This series converges only if $x=0$.



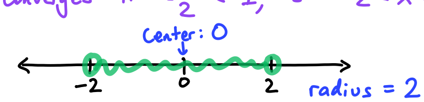
(c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n\,2^n}$ $\qquad a_n = \dfrac{x^n}{n\cdot 2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{n+1}}^{\,x}}{(n+1)\cancel{2^{n+1}}_{\,2}}\cdot\frac{n\cdot\cancel{2^n}}{\cancel{x^n}}\right| = \lim_{n\to\infty}\left|\frac{n\cdot x}{(n+1)\cdot 2}\right| = \frac{|x|}{2}$$

By the ratio test, this series converges if $\frac{|x|}{2}<1$, so $-2<x<2$.

Interval of convergence:

$-2 < x < 2$



7. **Experiment!** Go back to #6(a) and plug in $x=1$. Pull out a calculator/laptop and add up a TON of the terms. Discuss.

Try this for yourself!