Series: Comparison and Ratios

1. *None* of the series below is a geometric series or a p-series. However, ech one is *roughly equal* to a series that *is* a geometric series or a p-series. For each series below, find a geometric series or p-series that you think is "roughly the same?" as the given series.

☞ HINT: Think about what is truly "dominating" each numerator and each denominator when n is really, really huge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^2}$

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n+n}{3^n}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1+3^n}{2^n+n^{25}}$

Now, based on your intuition, can you figure out which of the series above converge and which diverge?

2. Look at the series $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots$

(a) Quick! Calculate the ratios: $\dfrac{a_1}{a_0}, \dfrac{a_2}{a_1}, \dfrac{a_3}{a_2}, \dfrac{a_4}{a_3}, \dfrac{a_5}{a_4}, \dfrac{a_6}{a_5}$.

(b) What is $\displaystyle\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)$?

3. Let's keep finding ratios of *consecutive terms* of a series! Now the series is

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{3} + 2\cdot\frac{4}{9} + 3\cdot\frac{8}{27} + 4\cdot\frac{16}{81} + 5\cdot\frac{32}{243} + \cdots.$$

What is a formula for $\dfrac{a_{n+1}}{a_n}$? What is $\displaystyle\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)$?

4. Milo: I am bummed, Delphine. The series $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ is *not* a geometric series. I don't know what to do!

Delphine: It's not so bad, Milo! The series $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ *behaves a lot like* the geometric series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$.

Group discussion: What do you think Delphine means by "behaves a lot like?" Do you think $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges or diverges?

5. Group discussion/intuition: For what values of r do you think $\sum_{n=1}^{\infty} n \cdot r^n$ converges? ☞ You just did the problem for $r = \frac{2}{3}$.

6. Let's try to use the ratio test to figure out whether $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

(a) Write down some formulas, then simplify the ratio as much as possible:

$a_n =$ $\qquad\qquad$ $a_{n+1} =$

$$\left|\frac{a_{n+1}}{a_n}\right| =$$

(b) Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$$

(c) What does the ratio test tells you for the series $\sum_{n=0}^{\infty} \frac{1}{n!}$?

7. Try to use the ratio test to determine whether each series converges.

(a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

(d) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$