MATH 126 — 13 Oct. 2025

RECALL: Geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ when $|r|<1$
diverges when $|r| \geq 1$

p-series $\sum_{n=1}^{\infty} \frac{a}{n^p}$ converges when $p>1$, diverges when $p \leq 1$

In particular, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ DIVERGES (Harmonic series)

while $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ CONVERGES

LIMIT COMPARISON TEST: If $a_n \approx b_n$ as $n \to \infty$,
meaning that $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $0 < c < \infty$,
then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

RATIO TEST: Calculate $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$.

- If $0 \leq r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then this test is inconclusive

Example: Using the limit comparison test:

Known series: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Unknown series: $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2-1}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2-1}} \cdot \frac{n^2-1}{n^2-1} = \lim_{n\to\infty} \frac{n^2-1}{n^2} = 1$$

Since the limit is 1 and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ also converges

#1(b) on the next page

Series: Comparison and Ratios

1. *None* of the series below is a geometric series or a p-series. However, ech one is *roughly equal* to a series that *is* a geometric series or a p-series. For each series below, find a geometric series or p-series that you think is "roughly the same?" as the given series.

👈 HINT: Think about what is truly "dominating" each numerator and each denominator when n is really, really huge.

When n is huge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\qquad \dfrac{1}{n^2+1} \approx \dfrac{1}{n^2}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ $\qquad \dfrac{1}{n^2-1} \approx \dfrac{1}{n^2}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ $\qquad \dfrac{1}{n^2-n} \approx \dfrac{1}{n^2}$

These series behave like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges.

diverges (d) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^2}$ $\quad$ Similar to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges

$\displaystyle = \sum_{n=2}^{\infty}\left(\frac{n}{n^2}+\frac{1}{n^2}\right) = \underbrace{\sum_{n=2}^{\infty}\frac{1}{n}}_{\text{diverges}} + \sum_{n=2}^{\infty}\frac{1}{n^2}$ so sum diverges

converges (e) $\displaystyle\sum_{n=2}^{\infty} \frac{2^n+n}{3^n}$ $\quad$ Similar to $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^n$, which is a convergent geometric series $\left(r=\frac{2}{3}\right)$

diverges (f) $\displaystyle\sum_{n=2}^{\infty} \frac{1+3^n}{2^n+n^{25}}$ $\quad$ Similar to $\displaystyle\sum_{n=2}^{\infty}\left(\frac{3}{2}\right)^n$, which is a divergent geometric series

Divergence Test: $\displaystyle\lim_{n\to\infty} \frac{1+3^n}{2^n+n^{25}} = \underset{(\infty)}{\text{DNE}} \neq 0$, so this series diverges!

Now, based on your intuition, can you figure out which of the series above converge and which diverge?

2. Look at the series $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}+\cdots$

(a) Quick! Calculate the ratios: $\dfrac{a_1}{a_0}, \dfrac{a_2}{a_1}, \dfrac{a_3}{a_2}, \dfrac{a_4}{a_3}, \dfrac{a_5}{a_4}, \dfrac{a_6}{a_5}$.

(b) What is $\displaystyle\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$? $\qquad \dfrac{a_{n+1}}{a_n} = \dfrac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n} = \dfrac{2}{3}$

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\frac{2}{3} = \frac{2}{3}$$

3. Let's keep finding ratios of *consecutive terms* of a series! Now the series is

not a geometric series!

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{3}+2\cdot\frac{4}{9}+3\cdot\frac{8}{27}+4\cdot\frac{16}{81}+5\cdot\frac{32}{243}+\cdots.$$

What is a formula for $\dfrac{a_{n+1}}{a_n}$? What is $\displaystyle\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$?

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)\cdot\left(\frac{2}{3}\right)^{n+1}}{n\cdot\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty}\frac{n+1}{n}\left(\frac{2}{3}\right) = \frac{2}{3}$$

4. Milo: I am bummed, Delphine. The series $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ is *not* a geometric series. I don't know what to do!

Delphine: It's not so bad, Milo! The series $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ *behaves a lot like* the geometric series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$.

Group discussion: What do you think Delphine means by "behaves a lot like?" Do you think $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges or diverges?

In the long run, each term is very nearly $\frac{2}{3}$ times the previous term, so this series behaves like the geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

5. Group discussion/intuition: For what values of r do you think $\sum_{n=1}^{\infty} n\cdot r^n$ converges? You just did the problem for $r = \frac{2}{3}$.

$|r| < 1$

6. Let's try to use the ratio test to figure out whether $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

(a) Write down some formulas, then simplify the ratio as much as possible:

$$a_n = \frac{1}{n!} \qquad a_{n+1} = \frac{1}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)!}\cdot n!}{\frac{1}{n!}\cdot n!}\right| = \frac{n!}{(n+1)!} = \frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{(3)}\cancel{(2)}(1)}{(n+1)\cancel{(n)}\cancel{(n-1)}\cancel{/}\cdots\cancel{(3)}\cancel{(2)}(1)} = \frac{1}{n+1}$$

(b) Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

(c) What does the ratio test tells you for the series $\sum_{n=0}^{\infty} \frac{1}{n!}$?

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, this series converges!

7. Try to use the ratio test to determine whether each series converges.

(a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ $\quad a_n = \frac{2^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2}{n+1} = 0$$

Since this limit is less than 1, the series CONVERGES.

(b) $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ $\quad a_n = \frac{n^2\cdot 2^{n+1}}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^2 2^{n+2}}{3^{n+1}}}{\frac{n^2 2^{n+1}}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2 2^{n+2}}{3^{n+1}}\cdot\frac{3^n}{n^2\cdot 2^{n+1}} = \lim\frac{2(n+1)^2}{3n^2} = \frac{2}{3}$$

Since this limit is less than 1, the series CONVERGES

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ $\quad a_n = \frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\sqrt{n+1}}{n+2}}{\frac{\sqrt{n}}{n+1}}\right| = \lim_{n\to\infty}\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} = \lim_{n\to\infty}\frac{n\sqrt{n+1}}{n\sqrt{n}} = 1$$

Since this limit is equal to 1, the ratio test is INCONCLUSIVE for this series!

(d) $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ $\quad a_n = \frac{n!}{10^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{10^{n+1}}}{\frac{n!}{10^n}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\cdot\frac{10^n}{n!} = \lim_{n\to\infty}\frac{n+1}{10} = \text{DNE } (\infty)$$

Since this limit is larger than 1, the series DIVERGES.