MATH 126 A/B — 10 October 2025

Harmonic Series: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

The individual terms get closer and closer to zero, but the harmonic series DIVERGES!

INTEGRAL TEST: Suppose $f(x)$ is a positive, decreasing, continuous function and $a_n = f(n)$. Then:

$$\sum_{n=1}^{\infty} a_n \text{ converges if } \int_1^{\infty} f(x)\,dx \text{ converges}$$

and $$\sum_{n=1}^{\infty} a_n \text{ diverges if } \int_1^{\infty} f(x)\,dx \text{ diverges.}$$

This is the worksheet from last class

3. Write an example of a geometric series that converges to 5. See if you can write an example *different* from everyone else at your table.

The SUM of your series should be 5.

4. Which of the following series diverge? How do you know?

Think about what happens to the partial sums as you add more and more terms.

$$\sum_{n=0}^{\infty}(-2)^n =$$

$$\sum_{n=1}^{\infty} n =$$

$$\sum_{n=2}^{\infty} \frac{n+1}{n} =$$

$$\sum_{n=1}^{\infty} \frac{n+1}{3n} =$$

adds up to $\frac{1}{2}$

adds up to $\frac{1}{2}$

adds up to $\frac{1}{2}$

5. **Erez:** I really like this series, but I don't know if it converges.

etc.

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right)+\frac{1}{32}+\cdots$$

Group Chat: Help Erez out. Does the series converge?

Hint: Do not find all the partial sums. Focus on s_1, s_2, s_4, s_8, s_{16}, etc.

This series is like $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

So it <u>DIVERGES</u>

Partial Sums:

$S_1 = 1$, $S_2 = 1.5$, $S_4 = 2$,

$S_8 = 2.5$, $S_{16} = 3$, $S_{32} = 3.5, \ldots$

Ava: Since you figured out Erez's series, can you tell me whether my series converges or diverges?

larger than $\frac{1}{2}$ larger than $\frac{1}{2}$ larger than $\frac{1}{2}$

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\frac{1}{17}+\cdots$$

This is the **HARMONIC SERIES** $\sum_{n=1}^{\infty}\frac{1}{n}$

Group chat: How can you apply what you learned about Erez's series to answer Ava's question?

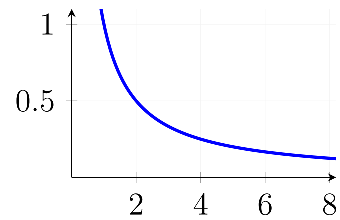
Partial sums of Ava's series are <u>larger</u> than the partial sums of Erez's series.

Since Erez's series diverges, Ava's series must also <u>DIVERGE</u>!

Series of Real Numbers

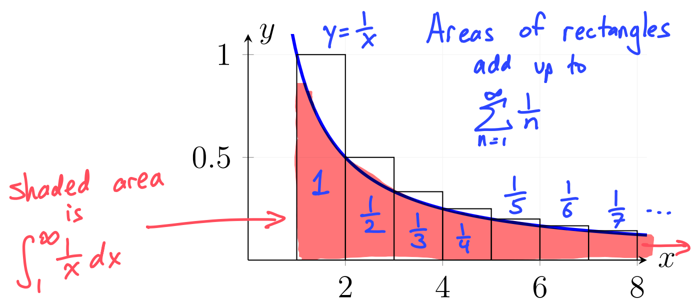
1. Lana: Hey, Simon! Want to draw with me?

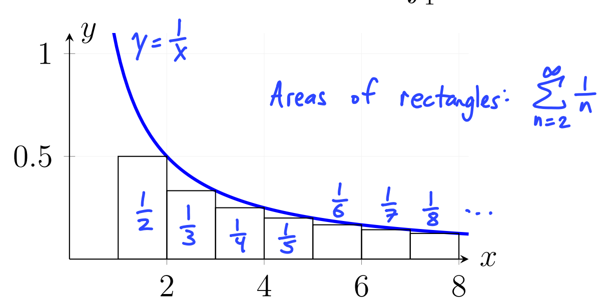
Simon: Sure, but I really miss integrals. Can we please go back? I'll draw $f(x) = \dfrac{1}{x}$.



Lana: YAY! This reminds me of when we did improper integrals.

Simon: I'll draw the left and right Riemann sum rectangles to approximate $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.





Lana: Now we just have to add up the areas of all of the rectangles!

Simon: Um....I thought I told you I wanted to go back to integrals...

Group Task: Write the left and right rectangle areas as sums that goes on forever.

☝ They are slightly different!

$$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\Big[\ln(x)\Big]_1^b = \lim_{b\to\infty}\Big[\ln(b) - \cancel{\ln(1)}^{\,0}\Big] = \lim_{b\to\infty}\ln(b)$$ DIVERGES (to ∞)

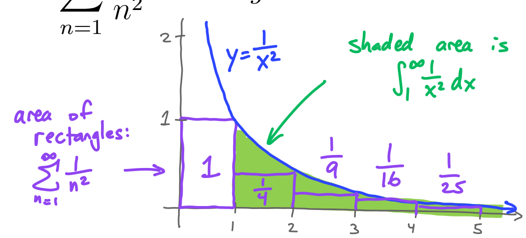
Simon: We know $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges, so the area underneath $y = \dfrac{1}{x}$ is infinite.

Lana: I guess that this must mean that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ *also* diverges!

Group chat: Why did Lana say this?

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges and $\sum_{n=1}^{\infty}\frac{1}{n} > \int_1^\infty \frac{1}{x}\,dx$, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ must also DIVERGE.

Simon: I think we can use the same reasoning to argue that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ *converges.* Amazing!



Group chat: Why does Simon think that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges?

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges and $\sum_{n=1}^{\infty}\frac{1}{n^2} < 1 + \int_1^\infty \frac{1}{x^2}\,dx$, the series must also converge.

Group chat: For which value(s) of p do you think $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges?

☝ Take a wild guess. I bet you're right!

This is called a p-series

Converges if $p > 1$.

Diverges if $p \le 1$.

We didn't get to this page in class, but here is one more example.

2. (a) Is the function $f(x) = \dfrac{2x}{(x^2+4)^2}$ continuous as $x \to \infty$? Yes!

(b) Is the function $f(x) = \dfrac{2x}{(x^2+4)^2}$ (at least eventually) positive as $x \to \infty$? Yes!

(c) Is the function $f(x) = \dfrac{2x}{(x^2+4)^2}$ (at least eventually) decreasing as $x \to \infty$? Yes!

☞ IF the answers to (a), (b), and (c) are all YES, you may TRY using the integral test.

(d) Does $\displaystyle\int_1^\infty \frac{2x}{(x^2+4)^2}\,dx$ converge or diverge? CONVERGE!

$u = x^2+4$, $du = 2x\,dx$

$$= \int_5^\infty \frac{1}{u^2}\,du = \lim_{b\to\infty} \int_5^b u^{-2}\,du = \lim_{b\to\infty} \left[\frac{-1}{u}\right]_5^b = \lim_{b\to\infty} \left[\frac{-1}{b} - \frac{-1}{5}\right] = \frac{1}{5}$$

(e) Does $\displaystyle\sum_{n=1}^\infty \frac{2n}{(n^2+4)^2}$ converge or diverge?

The integral test says that since $\displaystyle\int_1^\infty \frac{2x}{(x^2+4)}\,dx$ converges, so does $\displaystyle\sum_{n=1}^\infty \frac{2n}{(n^2+4)}$.

3. (a) If $n \geq 1$: $\dfrac{1}{n^2+n}$ is bigger/smaller than $\dfrac{1}{n^2}$.

☞ circle one

(b) The sum $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$ is bigger/smaller than the sum $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$.

(c) **Ava:** I really want to know if $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$ converges or diverges.

Milo: That series totally converges. Look at part (b) above!

Group Chat: Why do you think Milo is arguing that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$ converges?

4. (a) **Milo:** You can also figure out the answer to $\displaystyle\sum_{n=2}^\infty \frac{1}{n-\sqrt{n}}$ this way.

Group chat: What is Milo thinking? Does that series converge or diverge?

☞ Do you see why this series starts at $n = 2$?

(b) **Ava:** But Milo, your method is really picky! It *fails* to work for $\displaystyle\sum_{n=2}^\infty \frac{1}{n^2-1}$.

Milo: I guess so, but no worries! We know $\dfrac{1}{n^2-1} \approx \dfrac{1}{n^2}$, so it works out OK.

Group Chat: What are Milo and Ava talking about?