MATH 126 — 8 October 2025

LAST TIME: An infinite series looks like

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

To find the actual sum, we need the sequence of partial sums

$$S_n = a_1 + a_2 + \cdots + a_n.$$

Then, $\lim_{n\to\infty} S_n$ equals $\sum_{n=1}^{\infty} a_n$.

A geometric series is a series in which every term is the same multiple of the previous term.

A geometric series looks like this:

$$\underset{\substack{\uparrow \\ a\,=\,\text{initial term}}}{a} + \underset{\substack{\uparrow \\ r\,=\,\text{common ratio}}}{ar} + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$$

THEOREM: Geometric series $\sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$. → equivalently, $-1 < r < 1$

That is: If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

If $|r| \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges.

DIVERGENCE TEST: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ must diverge.

WARNING: This test can only rule out obviously-diverging series. It cannot tell you that a series converges.

Geometric Series, Convergence, and Divergence

1. Here is a geometric series: $\quad a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$

Consider a partial sum of this series:

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

multiply by r

(a) What is $r \cdot s_n$?

$$r \cdot s_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n+1}$$

(b) You might notice that s_n and $r \cdot s_n$ are VERY similar. What happens when you subtract them?

$$s_n - r \cdot s_n = a - ar^{n+1}$$

(c) Use the above to solve for s_n.

$$s_n(1-r) = a - ar^{n+1}$$

Look, you found a formula for s_n. WOW!

then: $\quad s_n = \dfrac{a - ar^{n+1}}{1-r} \qquad \boxed{s_n = a\,\dfrac{1-r^{n+1}}{1-r}}$

(d) In order for the geometric series to converge, you need $\lim_{n\to\infty} s_n$ to be a number (not infinity or oscillating). Try a few specific values for r. For what values of r does the limit exist?

Try positive, negative, big, small, etc.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a\,\dfrac{1-r^{n+1}}{1-r}$ exists if and only if $\lim_{n\to\infty} r^{n+1}$ exists,

which is the case if and only if $-1 < r < 1$.

another way to write this is $|r| < 1$

(e) If r is such that the geometric series converges, what does it converge TO?

If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} s_n = \frac{a}{1-r}$

2. Determine if the series converges or diverges. If the series converges, find the value that it converges to.

common ratio | initial term

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{-2}{3}\right)^n = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$ $\qquad r = \frac{-2}{3} \qquad a = \left(\frac{-2}{3}\right)^0 = 1$

Hint: Identify a and r.

(b) $\displaystyle\sum_{n=1}^{\infty} 4\left(\frac{1}{3}\right)^n = \frac{a}{1-r} = \frac{\frac{4}{3}}{1-\frac{1}{3}} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2$ $\qquad r = \frac{1}{3} \qquad a = 4\left(\frac{1}{3}\right)^1 = \frac{4}{3}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n+1}}{2^n} = \sum_{n=1}^{\infty} e\left(\frac{e}{2}\right)^n$ $\qquad r = \frac{e}{2} > 1 \qquad$ so this series diverges

recall: $e \approx 2.7\ldots$

3. Write an example of a geometric series that converges to 5. See if you can write an example *different* from everyone else at your table.

☜ The SUM of your series should be 5.

We didn't do this in class, but you want $\frac{a}{1-r} = 5$ and $|r| < 1$.

First choose any r such that $|r| < 1$. Then set $a = 5(1-r)$.

4. Which of the following series diverge? How do you know?

☜ Think about what happens to the partial sums as you add more and more terms.

$$\sum_{n=0}^{\infty}(-2)^n = 1 - 2 + 4 - 8 + 16 - 32 + \cdots$$ (geometric series with $r=-2$)

these terms get larger and larger, so this series can't converge!

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+6+\cdots$$ clearly this sum grows larger than any number you like!

$\lim_{n\to\infty} \frac{n+1}{n} = 1$

$$\sum_{n=2}^{\infty}\frac{n+1}{n} = \frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\frac{7}{6}+\frac{8}{7}+\cdots$$ These terms individually approach 1, so in the long run this series is like $1+1+1+1+1+\cdots$, which diverges!

$\lim_{n\to\infty} \frac{n+1}{3n} = \frac{1}{3}$

$$\sum_{n=1}^{\infty}\frac{n+1}{3n} = \frac{2}{3}+\frac{3}{6}+\frac{4}{9}+\frac{5}{12}+\frac{6}{15}+\frac{7}{18}+\cdots$$

These terms individually approach $\frac{1}{3}$, so in the long run this series is like $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\cdots$, which diverges

We will consider these next time!

5. Erez: I really like this series, but I don't know if it converges.

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\cdots$$

Group Chat: Help Erez out. Does the series converge?

☜ Hint: Do not find all the partial sums. Focus on s_1, s_2, s_4, s_8, s_{16}, etc.

Ava: Since you figured out Erez's series, can you tell me whether my series converges or diverges?

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\cdots$$

Group chat: How can you apply what you learned about Erez's series to answer Ava's question?