Introduction to Series

1. **Milo:** Hey Chloe, do you recall that the decimal 0.3 is the same as the fraction $\frac{3}{10}$?

Chloe: Yes! And, the decimal 0.33 is $\frac{33}{100}$. Isn't the base-ten number system great?

Milo: It sure is! Here's something fun: we can think about each digit in 0.33 *individually.* So, 0.33 can also be written as

$$0.33 = \frac{3}{10} + \frac{3}{100}.$$

Chloe: Good point. You can also write the numbers 0.333 and 0.3333 in a similar fashion.

Group chat: What does Chloe mean?

Chloe: We could even write the number $0.333\overline{3}$ in this way.

Group chat: Now what does Chloe mean? Write $0.333\overline{3}$ as a sum of fractions.

☞ The "bar" over the last 3 means it just keeps going ... for ever and ever and ever.

Milo: Nooooooooo...now we get a sum of *infinitely* many fractions.

Group chat: What's a formula for the nth individual number you see *within* the sum you wrote above?

Chloe: Wow! This is cool!

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \frac{3}{1000000} + \frac{3}{10000000} + \cdots \quad \text{EQUALS} \quad \frac{1}{3}$$

Group chat: Discuss Chloe's claim. Can you really add an infinite number of numbers?

2. Here is a fun sequence:

☞ Recall the word "sequence" from Friday's class!

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \ldots$$

Find a formula for the nth term of this sequence:

$$a_n =$$

Does this sequence converge or diverge?

3. (a) **Delphine:** What happens if we add up all of the numbers in that sequence in problem #2?

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\cdots$$

Sundar: Wow! Did you know that an infinitely long *sum* of numbers is called a **series**?

Easy group chat: Find a formula for nth term in this series.

(b) **Delphine:** I want to know if my series actually adds up to something!

Sundar: But you're adding up infinitely many numbers, which doesn't make sense to do.

Delphine: Except that in problem #1, Chloe added up infinitely many numbers and got an answer: $\frac{1}{3}$.

Sundar: Hmmm, you're right. I guess we should start adding numbers, then!

Delphine: Let's do this just like we did with improper integrals! We'll add *a few numbers at a time*!

Group chat: Why would adding a few numbers at a time be similar to what we did with improper integrals?

Delphine: I am going to create a new number and call it s_n:

☞ So, for example, s_3 is the sum of the first three numbers.

$$s_n = \text{the sum of the first } n \text{ numbers in the series}$$

Group calculation:

$s_1 =$ $s_2 =$ $s_3 =$ $s_4 =$ $s_5 =$

Group chat: Can you find a *formula* for the sum of the first n numbers?

$$s_n =$$

(c) **Delphine:** See? Now we can use our formula for s_n to find the sum of the *entire* series!

Group chat: What is the sum of the entire infinite series?

☞ Be ready to explain why you think this!

4. What do you think is the sum of each of the following?

☞ Can you come up with a formula that describes the n-th number in the sum?

(a) $2+2+2+2+2+2+2+2+2+\cdots$

(b) $2-2+2-2+2-2+2-2+2-2+\cdots$

5. In each of the following series, the next term is always the *same* multiple of the previous term. For each, find the "common ratio" between terms and write a summation formula starting with index $n = 0$.

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$

common ratio: $r =$ summation formula: $\sum_{n=0}^{\infty}$

(b) $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \frac{5}{243} + \cdots$

common ratio: $r =$ summation formula: $\sum_{n=0}^{\infty}$

(c) $-5 + \frac{5}{3} - \frac{5}{9} + \frac{5}{27} - \frac{5}{81} + \frac{5}{243} + \cdots$

common ratio: $r =$ summation formula: $\sum_{n=0}^{\infty}$

(d) $\frac{1}{3} + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots$

common ratio: $r =$ summation formula: $\sum_{n=0}^{\infty}$

6. These series also have the property that the next term is always the *same* multiple of the previous term. For each, find the first term. Then find the "common ratio" between terms.

Be careful: "first term" does not necessarily mean $n = 1$.

(a) $\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n$

(b) $\sum_{n=1}^{\infty} \frac{4}{3^n}$

(c) $\sum_{n=3}^{\infty} \frac{e^{n+1}}{\pi^n}$

(d) $\sum_{n=0}^{\infty} \frac{6^{n+1}}{5^{2n}}$