MATH 126 A/B — 6 October 2025

Recall: A sequence is a list of numbers.

New: A series is a sum of numbers.

An infinite series is an infinite sum of numbers.

DEFINITION OF CONVERGENCE FOR SERIES:

Let $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$ be an infinite sequence.

Form the sequence of partial sums:

$$
\begin{aligned}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n
\end{aligned}
$$

Then, the sum of the infinite series is $\lim_{n \to \infty} S_n$.

The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $\{S_n\}$ converges

Introduction to Series

1. Milo: Hey Chloe, do you recall that the decimal 0.3 is the same as the fraction $\frac{3}{10}$?

Chloe: Yes! And, the decimal 0.33 is $\frac{33}{100}$. Isn't the base-ten number system great?

Milo: It sure is! Here's something fun: we can think about each digit in 0.33 *individually*. So, 0.33 can also be written as

$$0.33 = \frac{3}{10} + \frac{3}{100}.$$

Chloe: Good point. You can also write the numbers 0.333 and 0.3333 in a similar fashion.

Group chat: What does Chloe mean?

$$0.333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$$

$$0.3333 = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$$

Chloe: We could even write the number $0.333\overline{3}$ in this way.

Group chat: Now what does Chloe mean? Write $0.333\overline{3}$ as a sum of fractions.

☞ The "bar" over the last 3 means it just keeps going ... for ever and ever and ever.

$$0.\overline{3} = \underbrace{\frac{3}{10}}_{n=1} + \underbrace{\frac{3}{100}}_{n=2} + \underbrace{\frac{3}{1000}}_{n=3} + \cdots$$

index:

Milo: Noooooooooo...now we get a sum of *infinitely* many fractions.

Group chat: What's a formula for the nth individual number you see *within* the sum you wrote above?

the nth term is $\frac{3}{10^n}$

Chloe: Wow! This is cool!

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \frac{3}{100000} + \frac{3}{1000000} + \frac{3}{10000000} + \cdots \quad \text{EQUALS} \quad \frac{1}{3}$$

Group chat: Discuss Chloe's claim. Can you really add an infinite number of numbers?

It seems that we can, in this case at least!

2. Here is a fun sequence:

☞ Recall the word "sequence" from Friday's class!

$$\underbrace{\frac{1}{2}}_{n=1}, \overset{n=2}{\frac{1}{4}}, \underbrace{\frac{1}{8}}_{n=3}, \overset{n=4}{\frac{1}{16}}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \ldots$$

index:

Find a formula for the nth term of this sequence:

$$a_n = \frac{1}{2^n}$$

Does this sequence converge or diverge?

$\lim_{n\to\infty} \frac{1}{2^n} = 0$, so this sequence converges

3. (a) **Delphine:** What happens if we add up all of the numbers in that sequence in problem #2?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \cdots$$

Sundar: Wow! Did you know that an infinitely long *sum* of numbers is called a **series**?

Easy group chat: Find a formula for nth term in this series.

same as on the previous page

$$a_n = \frac{1}{2^n}$$

(b) **Delphine:** I want to know if my series actually adds up to something!

Sundar: But you're adding up infinitely many numbers, which doesn't make sense to do.

Delphine: Except that in problem #1, Chloe added up infinitely many numbers and got an answer: $\frac{1}{3}$.

Sundar: Hmmm, you're right. I guess we should start adding numbers, then!

Delphine: Let's do this just like we did with improper integrals! We'll add *a few numbers at a time*!

Group chat: Why would adding a few numbers at a time be similar to what we did with improper integrals?

Accumulating area as the bound of integration goes to ∞ is similar to adding up more and more numbers in an infinite series.

Delphine: I am going to create a new number and call it s_n:

$$s_n = \text{the sum of the first } n \text{ numbers in the series}$$

☞ So, for example, s_3 is the sum of the first three numbers.

Group calculation:

$$s_1 = \frac{1}{2} \qquad s_2 = \frac{3}{4} \qquad s_3 = \frac{7}{8} \qquad s_4 = \frac{15}{16} \qquad s_5 = \frac{31}{32}$$

numerator is one less than the denominator

denominator is a power of 2

Group chat: Can you find a *formula* for the sum of the first n numbers?

$$s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

(c) **Delphine:** See? Now we can use our formula for s_n to find the sum of the *entire* series!

Group chat: What is the sum of the entire infinite series?

☞ Be ready to explain why you think this!

The infinite sum is: $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1$

4. What do you think is the sum of each of the following?

☞ Can you come up with a formula that describes the n-th number in the sum?

these series diverge!

(a) $2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + \cdots$ — partial sums increase toward ∞

(b) $\underbrace{\underbrace{\underbrace{\underbrace{2}_{2} - 2}_{0} + 2}_{2} - 2}_{0} + 2 - 2 + 2 - 2 + 2 - 2 + \cdots$

partial sums alternate: 2, 0, 2, 0, 2, 0, ...

5. In each of the following series, the next term is always the *same* multiple of the previous term. For each, find the "common ratio" between terms and write a summation formula starting with index $n = 0$.

all of these series are **GEOMETRIC SERIES**

(a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$

$n=0$, $n=1$, $n=2$, $n=3$

$a_n = \left(\frac{1}{3}\right)^n = \frac{1}{3^n}$

common ratio: $r = \frac{1}{3}$ summation formula: $\sum_{n=0}^{\infty} \frac{1}{3^n}$

(b) $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \frac{5}{243} + \cdots$

common ratio: $r = \frac{1}{3}$ summation formula: $\sum_{n=0}^{\infty} 5 \cdot \frac{1}{3^n}$

(c) $-5 + \frac{5}{3} - \frac{5}{9} + \frac{5}{27} - \frac{5}{81} + \frac{5}{243} + \cdots$

common ratio: $r = \frac{-1}{3}$ summation formula: $\sum_{n=0}^{\infty} (-5)\left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} 5 \frac{(-1)^{n+1}}{3^n}$

(d) $\frac{1}{\cancel{3}\,1} + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots$

common ratio: $r = \frac{2}{3}$ summation formula: $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

6. These series also have the property that the next term is always the *same* multiple of the previous term. For each, find the first term. Then find the "common ratio" between terms.

Be careful: "first term" does not necessarily mean $n = 1$.

	First term	Common ratio
(a) $\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n$	$a_0 = 1$	$r = \frac{-2}{3}$
(b) $\sum_{n=1}^{\infty} \frac{4}{3^n}$	$a_1 = \frac{4}{3}$	$r = \frac{1}{3}$
(c) $\sum_{n=3}^{\infty} \frac{e^{n+1}}{\pi^n}$	$a_3 = \frac{e^4}{\pi^3}$	$r = \frac{e}{\pi}$
(d) $\sum_{n=0}^{\infty} \frac{6^{n+1}}{5^{2n}}$	$a_0 = 6$	$r = \frac{6}{25}$

recall that $5^{2n} = (5^2)^n = 25^n$