MATH 126 A/B   3 October 2025

SEQUENCES: new idea, old theme (infinity, limits, functions)

A **sequence** is simply a list of numbers.
The individual numbers are the **terms** of the sequence.
(infinite sequence: infinitely many terms.)

EXAMPLE: $1, 2, 3, 4, 5, 6, \ldots$

notation: $a_1 \quad a_2 \quad a_3 \quad a_4 \quad \ldots$ formula: $a_n = n$

We denote the whole sequence as $\{a_n\}$ or $\{n\}$.

Alternatively, we could start the indexes at $n = 0$:

$a_0 = 1, \quad a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \ldots$

In this case, $a_n = n + 1$.

EXAMPLE: $1, 2, 6, 24, 120, 720, \ldots$

Here, $a_n = n!$, where $n!$ is the factorial of n,

$$n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1).$$

A note about limits: Logarithms, polynomials, exponentials, and factorials all grow towards ∞, but at different rates.

grow fastest

factorials: $n!$

exponentials: $2^n, e^n, 3^n, \ldots$

powers: $n, n^2, n^3, n^4, \ldots$

logarithms: $\ln(n)$

grow slowest

Sequences and Limits

1. Find a formula for the nth term of the following infinite sequences:

👈 Go ahead and assume we start with $n = 1$. It's not always possible to find a formula, but for these examples you can.

$$\{a_n\} = \left\{e \cdot \pi, \frac{e \cdot \pi^2}{2}, \frac{e \cdot \pi^3}{3}, \frac{e \cdot \pi^4}{4}, \frac{e \cdot \pi^5}{5}, \ldots\right\} \qquad a_n = \frac{e\pi^n}{n}$$

$$\{b_n\} = \left\{1, \tfrac{2}{3}, \tfrac{3}{5}, \tfrac{4}{7}, \tfrac{5}{9}, \tfrac{6}{11}, \ldots\right\} \qquad b_n = \frac{n}{2n-1}$$

$$\{c_n\} = \left\{1, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{1}{24}, \tfrac{1}{120}, \tfrac{1}{720}, \ldots\right\} \qquad c_n = \frac{1}{n!}$$

$$\{d_n\} = \{-1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots\} \qquad d_n = (-1)^n$$

$$\{j_n\} = \left\{-1, \tfrac{1}{2}, \tfrac{-1}{6}, \tfrac{1}{24}, \tfrac{-1}{120}, \tfrac{1}{720}, \ldots\right\} \qquad j_n = \frac{(-1)^n}{n!}$$

$$\{k_n\} = \{3, 7, 3, 7, 3, 7, \ldots\} \qquad k_n = 5 + 2(-1)^n$$

2. For the sequences above...

(a) What is $\lim_{n\to\infty} a_n$? does not exist

(b) What is $\lim_{n\to\infty} b_n$? $\frac{1}{2}$

(c) What is $\lim_{n\to\infty} c_n$? 0

(d) What is $\lim_{n\to\infty} d_n$? does not exist

(e) What is $\lim_{n\to\infty} j_n$? 0

(f) What is $\lim_{n\to\infty} k_n$? does not exist

3. **Erez:** This isn't calculus! I miss integrals. Improper integrals were pretty interesting.

Chloe: Agreed. But did you know there are similarities at least? We still get to say the words "converge" and "diverge."

Erez: We have not defined those words in the context of sequences!

Chloe: Oh, but we can! The definition is not really any different!

Group chat: What does Chloe mean? How would you define converge and diverge in the context of sequences?

converge: the terms get closer and closer to some specific value

diverge: not converge

4. Look again at the sequences at the top of this page. Which sequences converge? Which diverge?

5. For each of the following formulas for a_n, does the sequence $\{a_n\}$ converge or diverge? If it converges, what does it converge to?

(a) $a_n = \dfrac{27}{n^2-2}$ $\{a_n\}$ converges to 0

(b) $a_n = \dfrac{1n^3+4n^2+1}{3n^3-2n+4}$ $\{a_n\}$ converges to $\frac{1}{3}$

(c) $a_n = \dfrac{n^3+4n^2+1}{3n^4-2n+4}$ $\{a_n\}$ converges to 0

(d) $a_n = \sin(\pi n)$ $a_1=0,\ a_2=0,\ a_3=0,\ a_4=0,\ldots$ All terms are 0, so this sequence converges to 0

👈 Careful! ☺

(e) $a_n = \cos(\pi n)$ $a_1=\cos(\pi)=-1,\ a_2=\cos(2\pi)=1,\ a_3=\cos(3\pi)=-1,\ a_4=\cos(4\pi)=1,\ldots$ This sequence oscillates, so it diverges.

(f) $a_n = \dfrac{n^7}{2^n}$ The denominator grows much faster than the numerator, so $\{a_n\}$ converges to 0.

6. Phil: Hey there, Renita! Have I shown you my favorite sequence? Here it is:

$$2.9, 2.95, 2.959, 2.9595, 2.95959, 2.959595, \ldots$$

Renita: Oh, I get it! I love patterns.

Phil: The problem is I am having trouble finding a formula for it, so I cannot figure out how to argue that this sequence converges, even though it clearly does!

Renita: The terms of your sequence are getting larger and larger, right?

Phil: You betcha! ↳ the sequence is strictly increasing

Renita: And none of the terms of your sequence are larger than 3, right?

Phil: Yeah... ↳ the sequence is bounded (above)

Renita: That's enough information to guarantee your sequence converges!

Group chat: Why does Renita say "that's enough information to guarantee your sequence converges"?

Theorem: A sequence that is strictly increasing and bounded must converge.

7. Try to modify Renita's argument for the sequence

$$2.1, 2.01, 2.001, 2.0001, 2.00001, 2.000001, 2.0000001, \ldots$$

This sequence is strictly decreasing and bounded (below)

Theorem: A sequence that is strictly decreasing and bounded (below) must converge.