Improper Integrals

1. (a) **Graphing time!** Choose someone in your group to sketch the graph of $y = \dfrac{1}{x^2}$ on the wall near your table.

(b) **Warm-up:** Use the FTC to compute these progressively fun areas.

☞ Draw the regions! Write out your answers as decimals numbers.

$$\int_1^{10} \frac{1}{x^2}\,dx \qquad \int_1^{100} \frac{1}{x^2}\,dx \qquad \int_1^{1000} \frac{1}{x^2}\,dx$$

$$\int_1^{10} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{10} = \frac{-1}{10} - \frac{-1}{1} = 1 - \frac{1}{10} = 0.9$$

$$\int_1^{100} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{100} = \frac{-1}{100} - \frac{-1}{1} = 1 - \frac{1}{100} = 0.99$$

$$\int_1^{1000} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{1000} = \frac{-1}{1000} - \frac{-1}{1} = 1 - \frac{1}{1000} = 0.999$$

(c) Now use the FTC to find a formula for $\displaystyle\int_1^{b} \frac{1}{x^2}\,dx$.

☞ You can pretend that b is some number larger than 1.

$$\int_1^{b} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^{b} = \frac{-1}{b} - \frac{-1}{1} = 1 - \frac{1}{b}$$

(d) **Milo:** Great! That must mean that $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ equals 1.

Erez: Wait...what? Infinity *in an integral*? You can't just replace b with ∞ like that!

Ava: Yeah, ∞ is not a number.

Milo: I know it looks awful, but it works!

Group chat: What is Milo thinking? Can you make sense of $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$?

As b gets larger and larger, $\int_1^b \frac{1}{x^2}\,dx$ becomes something like $\int_1^\infty \frac{1}{x^2}\,dx$.

Also as b gets larger and larger, $\int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}$ gets closer and closer to 1.

Thus, it makes sense that $\int_1^\infty \frac{1}{x^2}\,dx = \lim\limits_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = 1$.

2. (a) Now use the FTC to compute the following:

$$\int_{1/10}^{1} \frac{1}{x^2}\,dx \qquad \int_{1/100}^{1} \frac{1}{x^2}\,dx \qquad \int_{1/1000}^{1} \frac{1}{x^2}\,dx$$

$$\int_{1/10}^{1} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{1/10}^{1} = \frac{-1}{1} - \frac{-1}{\frac{1}{10}} = -1 + 10 = 9$$

$$\int_{1/100}^{1} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{1/100}^{1} = \frac{-1}{1} - \frac{-1}{\frac{1}{100}} = -1 + 100 = 99$$

$$\int_{1/1000}^{1} \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{1/1000}^{1} = \frac{-1}{1} - \frac{-1}{\frac{1}{1000}} = -1 + 1000 = 999$$

(b) **Group chat:** Now try to find a formula for $\int_a^1 \frac{1}{x^2}\,dx$.

Your formula will depend on a. You may suppose $0 < a < 1$.

$$\int_a^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_a^1 = \frac{-1}{1} - \frac{-1}{a} = \frac{1}{a} - 1$$

(c) **Ava:** Ooooh, now I want to calculate $\int_0^1 \frac{1}{x^2}\,dx$.

Erez (shaking head): You can't do that, because $\frac{1}{x^2}$ is not defined at $x = 0$.

Ava: True, but the integral still makes sense!

Group chat: Does $\int_0^1 \frac{1}{x^2}\,dx$ make sense? What happens to $\int_a^1 \frac{1}{x^2}\,dx$ as a gets closer and closer to zero?

$\int_0^1 \frac{1}{x^2}\,dx$ might make sense as the area under the graph of $y = \frac{1}{x^2}$ between $x=0$ and $x=1$.

However, as a gets closer and closer to zero, $\int_a^1 \frac{1}{x^2}\,dx$ gets larger and larger (without bound), so it seems the area under $y = \frac{1}{x^2}$ between $x=0$ and $x=1$ is <u>infinite</u>.

3. (a) Find a formula for $\int_a^1 \frac{1}{x}\,dx$.

☞ Again, suppose $0 < a < 1$.

$$\int_a^1 \frac{1}{x}\,dx = \ln|x|\Big|_a^1 = \ln(1) - \ln(a) = 0 - \ln(a)$$

(b) How should we evaluate $\int_0^1 \frac{1}{x}\,dx$?

$$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0} \int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0} \big(-\ln(a)\big) \text{ does not exist } (\infty)$$

So $\int_0^1 \frac{1}{x}\,dx$ <u>diverges</u>.

(c) How should we evaluate $\int_{-1}^0 \frac{1}{x}\,dx$?

(d) **Group chat:** How should we evaluate $\int_{-1}^1 \frac{1}{x}\,dx$?

$$\int_{-1}^1 \frac{1}{x}\,dx = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$$

4. Evaluate $\int_0^4 \frac{1}{\sqrt{x}}\,dx$. Either determine the number that it converges to, or explain why it diverges.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0} \int_a^4 x^{-1/2}\,dx = \lim_{a\to 0} \left[2x^{1/2}\right]_a^4 = \lim_{a\to 0} \left[2\sqrt{4} - 2\sqrt{a}\right]$$

$$= \lim_{a\to 0} \left[4 - 2\sqrt{a}\right] = 4$$

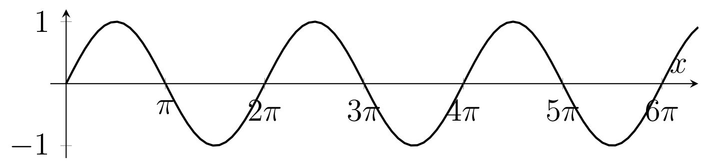
5. Evaluate $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$. Either determine the number that it converges to, or explain why it diverges.

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_1^b x^{-1/2}\,dx = \lim_{b\to\infty} \left[2x^{1/2}\right]_1^b = \lim_{b\to\infty} \left[2\sqrt{b} - 2\sqrt{1}\right]$$

$$= \lim_{b\to\infty} \left[2\sqrt{b} - 2\right] \text{ does not exist } (\infty)$$

So the integral <u>diverges</u>.

6. Here is a graph of the function $f(x) = \sin(x)$:



(a) Quick! Without calculating any antiderivatives or doing any algebra, find each of the following:

$$\int_0^{2\pi} \sin(x)\,dx = 0 \qquad \int_0^{4\pi} \sin(x)\,dx = 0 \qquad \int_0^{6\pi} \sin(x)\,dx = 0$$

(b) **Group chat:** Based on the previous question, what do you think $\int_0^{\infty} \sin(x)\,dx$ should equal? 0

(c) Use the FTC to evaluate $\int_0^{\pi} \sin(x)\,dx$.

$$\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi) - -\cos(0) = 2$$

☜ This is the area of one "bump" of the graph of $\sin(x)$.

(d) Without calculating any more antiderivatives or doing any more algebra, find each of the following:

$$\int_0^{3\pi} \sin(x)\,dx = 2 \qquad \int_0^{5\pi} \sin(x)\,dx = 2 \qquad \int_0^{7\pi} \sin(x)\,dx = 2$$

(e) **Group chat:** Based on the previous question, what do you think $\int_0^{\infty} \sin(x)\,dx$ should equal? 2

(f) **Group chat:** Considering parts (b) and (e) above, what can we say about $\int_0^{\infty} \sin(x)\,dx$?

This improper integral must diverge, since it can't equal both 0 and 2.

7. Experiment time! Choose a few different values of p and compute $\int_1^{\infty} \frac{1}{x^p}\,dx$. For which values of p do you get an actual numerical answer?