MATH 126 A/B — 1 October 2025

QUESTION: How do we make sense of $\int_1^\infty \frac{1}{x^2}\,dx$?

Two-step process:

$$\lim_{b\to\infty}\Bigg(\underbrace{\int_1^b \frac{1}{x^2}\,dx}_{\text{First, calculate this using FTC}}\Bigg)$$

Then, let $b \to \infty$.

$$\lim_{b\to\infty}\left(\int_1^b \frac{1}{x^2}\,dx\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$$

goes to zero as $b\to\infty$

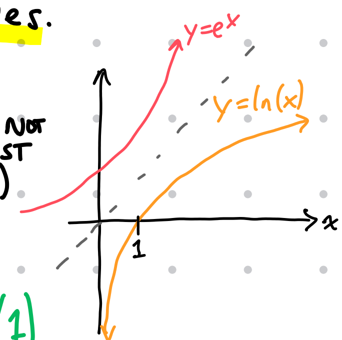
DEFINITION: $\displaystyle\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$

- If the limit exists, then the original integral converges.
- If the limit does not exist, then the original integral diverges.

EXAMPLE: $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\Bigg(\underbrace{\int_1^b \frac{1}{x}\,dx}_{\text{do this first}}\Bigg) = \lim_{b\to\infty}\big(\ln(b)\big) =$ DOES NOT EXIST (∞)

So, this integral diverges.

$$\int_1^b \frac{1}{x}\,dx = \ln|x|\Big|_1^b = \ln(b) - \ln(1) = \ln(b)$$



TERMINOLOGY: Any integral is called improper if either:

- A bound is $\pm\infty$.
- The function has a vertical asymptote either at a bound or somewhere between the bounds.

EXAMPLE: Function with a vertical asymptote at a bound of integration:

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left(\int_a^1 \frac{1}{x^2}\,dx\right) = \lim_{a\to 0^+}\left(\frac{1}{a}-1\right) = \text{does not exist } (\infty)$$

improper integral
DIVERGES

do this first

$$\int_a^1 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_a^1 = \frac{-1}{1} - \frac{-1}{a} = \frac{1}{a} - 1$$

CAUTION: If a function has a vertical asymptote <u>between</u> the bounds of integration, you must split up the integral into a sum of several improper integrals.

EXAMPLE: $f(x) = \frac{1}{x}$ has a vertical asymptote at $x=0$.

Thus:

$$\int_{-1}^1 \frac{1}{x}\,dx = \underbrace{\int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx}_{\text{both of these integrals diverge}}$$

so this integral <u>also</u> diverges